Well-balanced schemes for the Euler equations with gravitation

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Outline

- **Introduction**
- **Well-balanced scheme for HydroStatic Equilibrium (HSE)**
  - First order
  - Second order
- **Multi-D & further extensions**
- **Conclusions**

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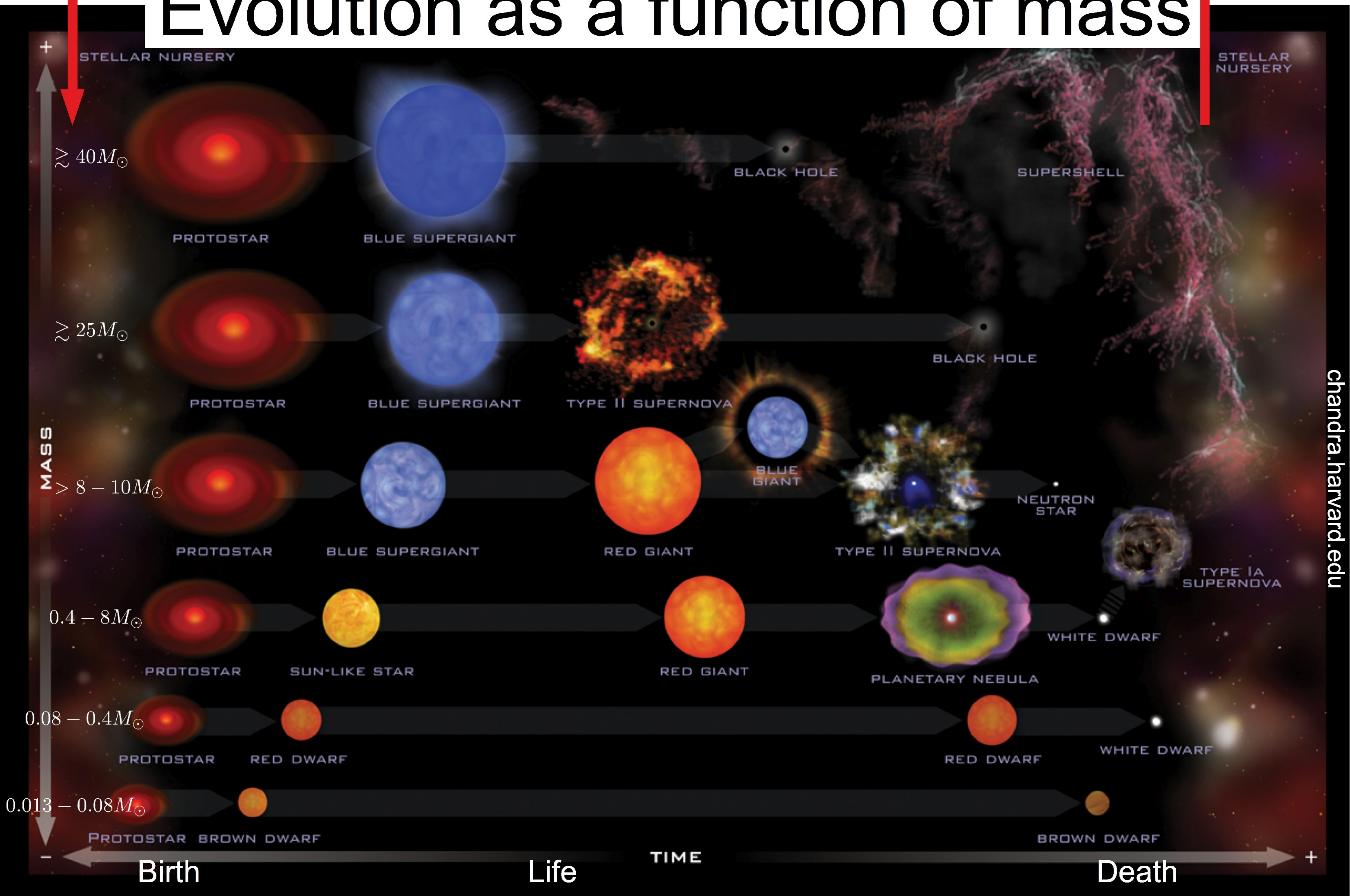
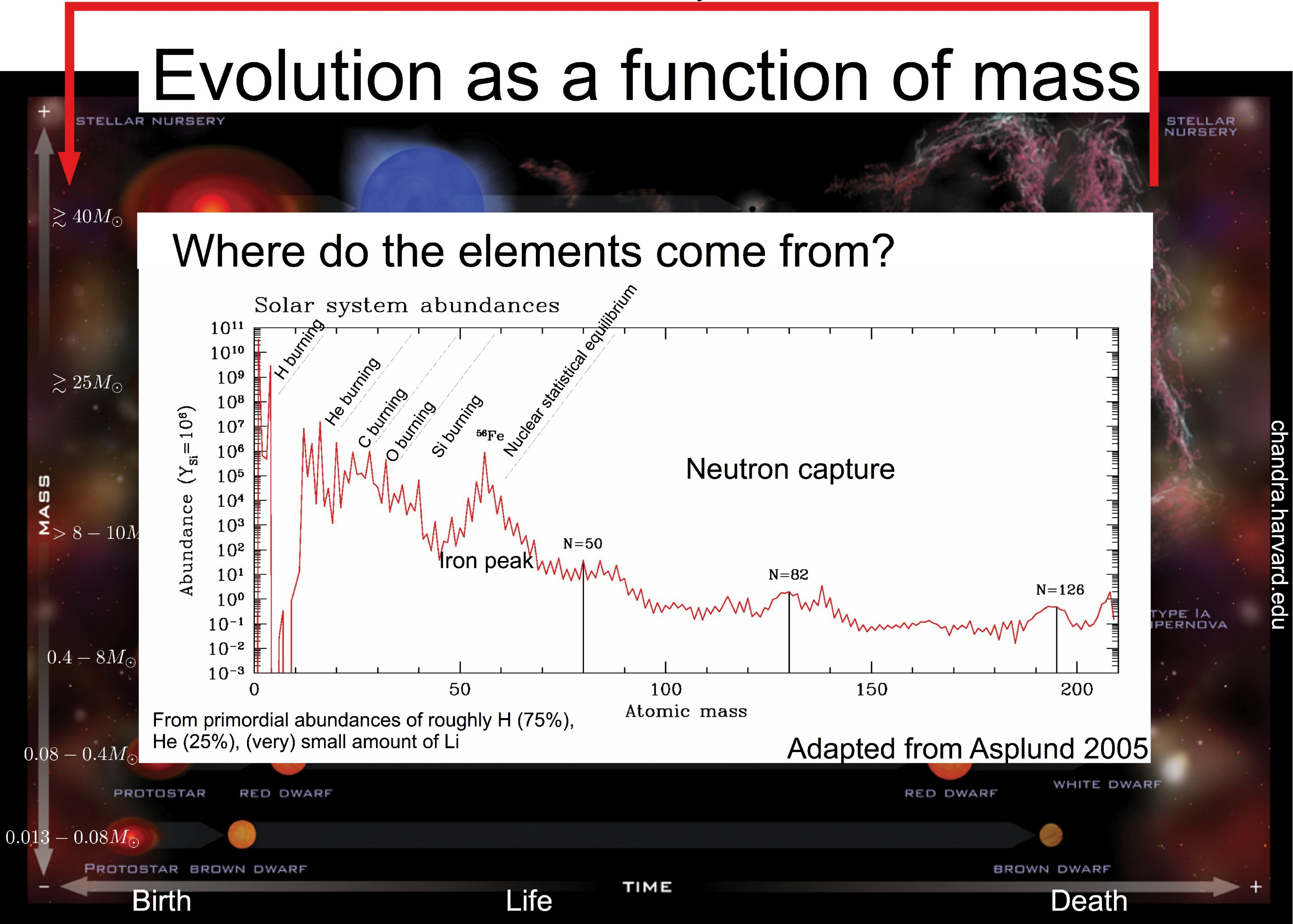
Stellar life cycle

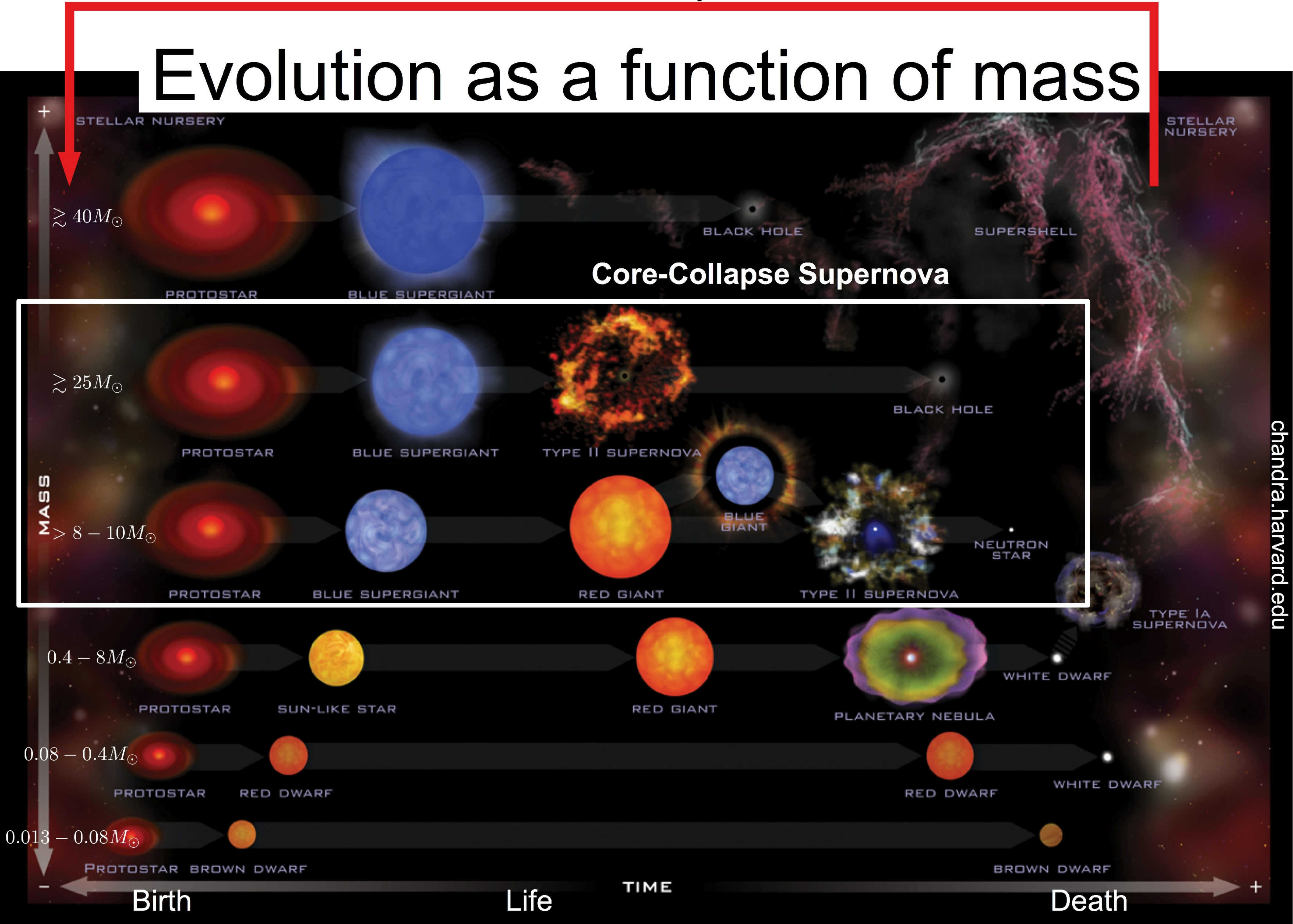
Evolution as a function of mass



Stellar life cycle



Stellar life cycle



Core-collapse supernova

- General idea:
  - Implosion of iron core of massive $M \gtrsim 8M_{\odot}$ at the end of thermonuclear evolution
  - Explosion powered by gravitational binding energy of forming compact remnant:

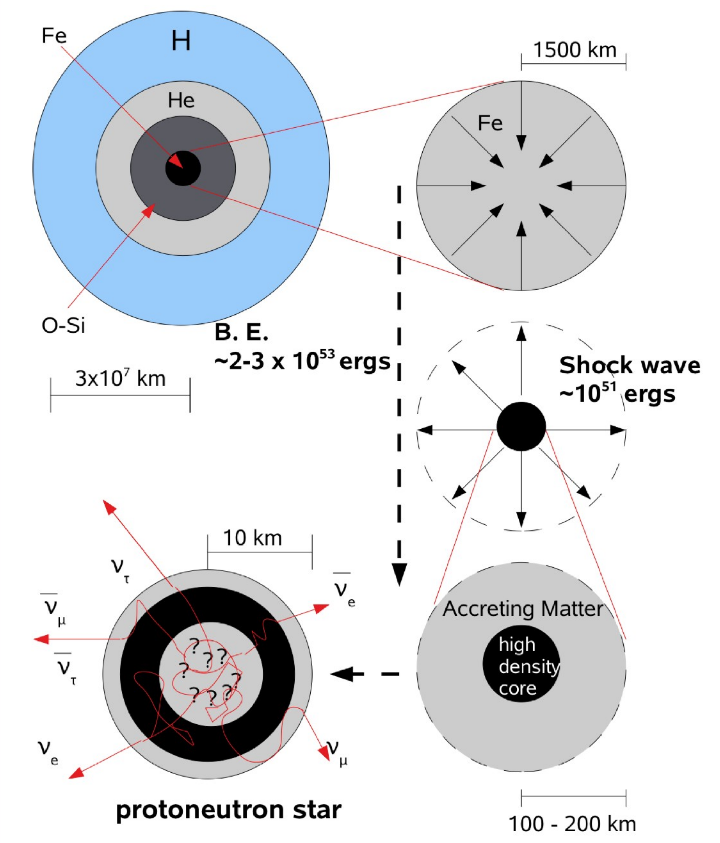
$$E_b \approx 3 \times 10^{53} \left(\frac{M}{M_{\odot}}\right)^2 \left(\frac{R}{10\mathrm{km}}\right)^{-1} \mathrm{erg}$$

GRAVITY BOMB!

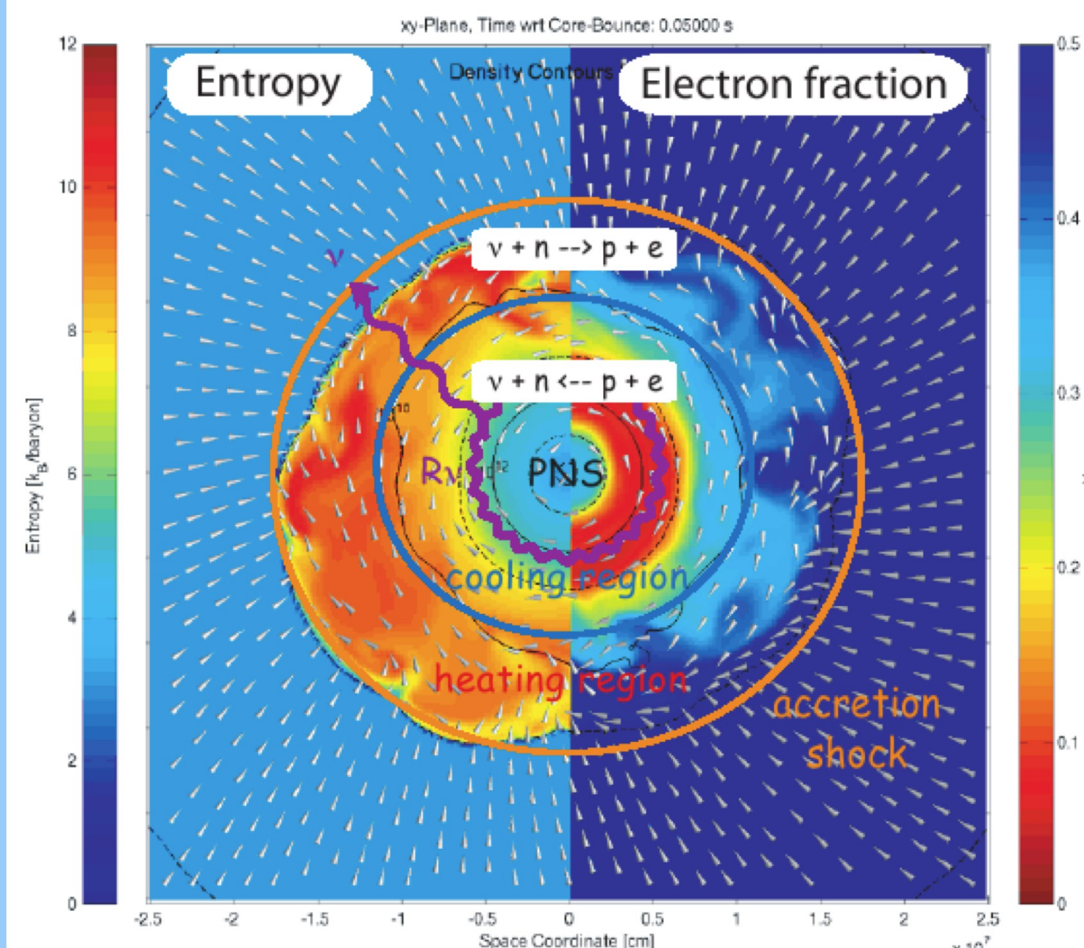
M Mass of remnant

R Radius of remnant

Core-collapse supernova



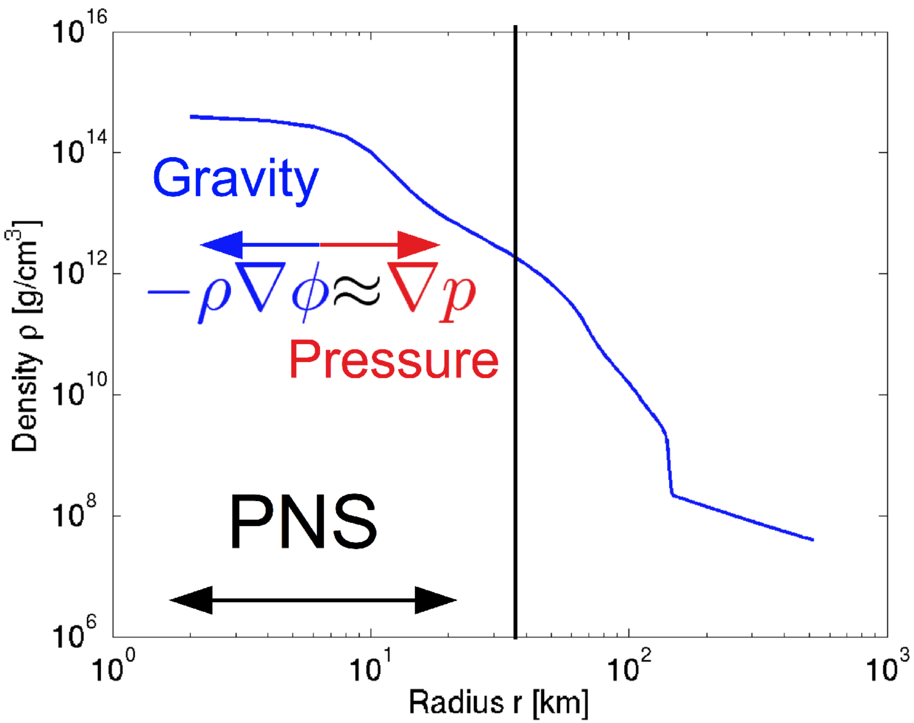
From S. Scheidegger

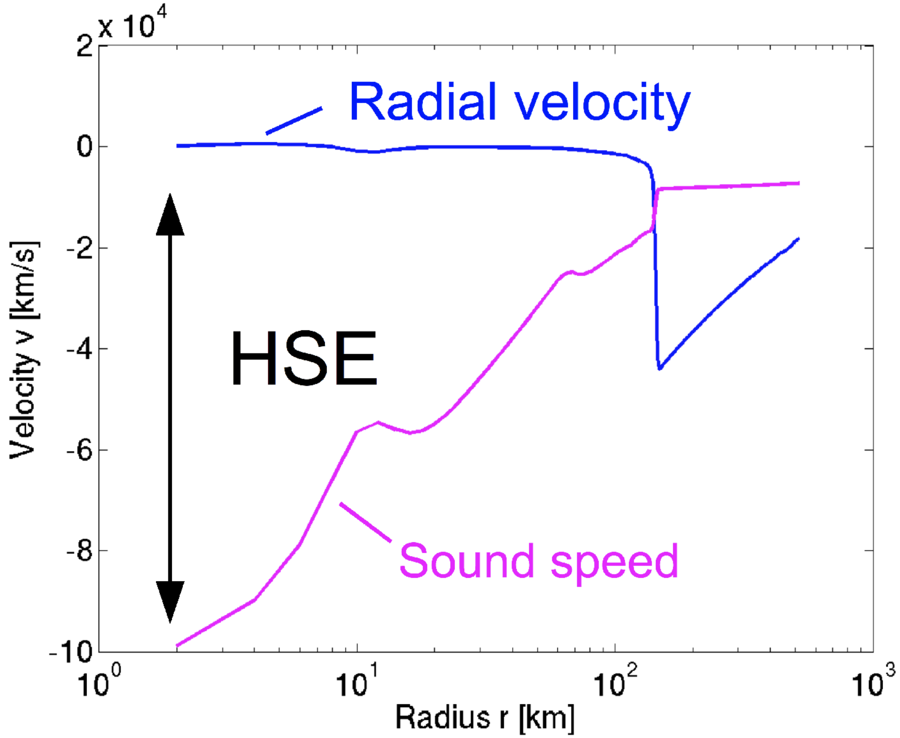


From M. Liebendoerfer

Radial profile

- The problem: (in our simulations)





Ability to maintain near hydrostatic equilibrium for a long time!

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Well-balanced scheme for HSE

- Consider 1D hydrodynamics eqs with gravity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S}$$

$$\boldsymbol{u} = \left[\begin{array}{c} \rho \\ \rho v \\ E \end{array} \right] \quad \boldsymbol{F} = \left[\begin{array}{c} \rho v \\ \rho v^2 + p \\ (E+p)v \end{array} \right] \quad \boldsymbol{S} = - \left[\begin{array}{c} 0 \\ \rho \\ \rho v \end{array} \right] \frac{\partial \phi}{\partial x}$$

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)

Well-balanced scheme for HSE (2)

- Classical solution algorithm:

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right) + \Delta t \boldsymbol{S}_i^n$$

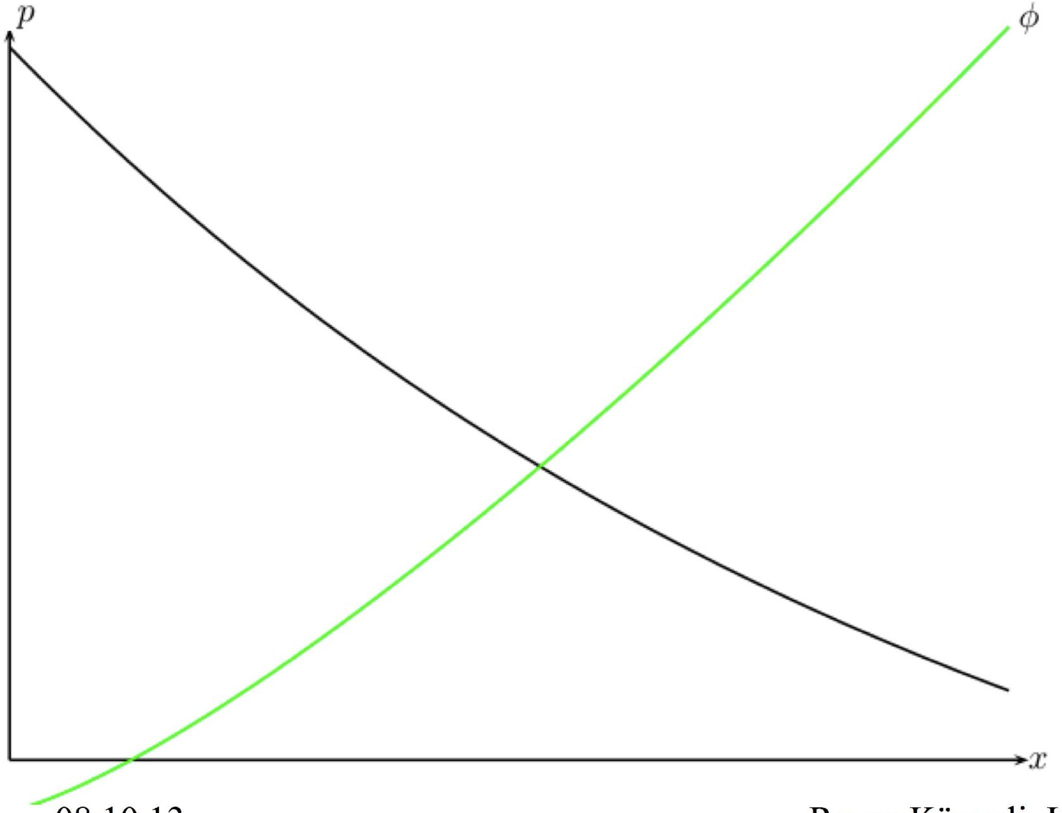
- Numerical flux $\boldsymbol{F}_{i\pm1/2}^n = \boldsymbol{\mathcal{F}}(\boldsymbol{u}_{i\pm1/2}^{n,L}, \boldsymbol{u}_{i\pm1/2}^{n,R})$ from (approximate) Riemann solver, e.g.
  - (Local) Lax-Friedrichs Lax (1954), Rusanov (1961)
  - HLL (C) Harten, Lax and van Leer (1983), Toro et al. (1994)
  - Roe Roe (1981)

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



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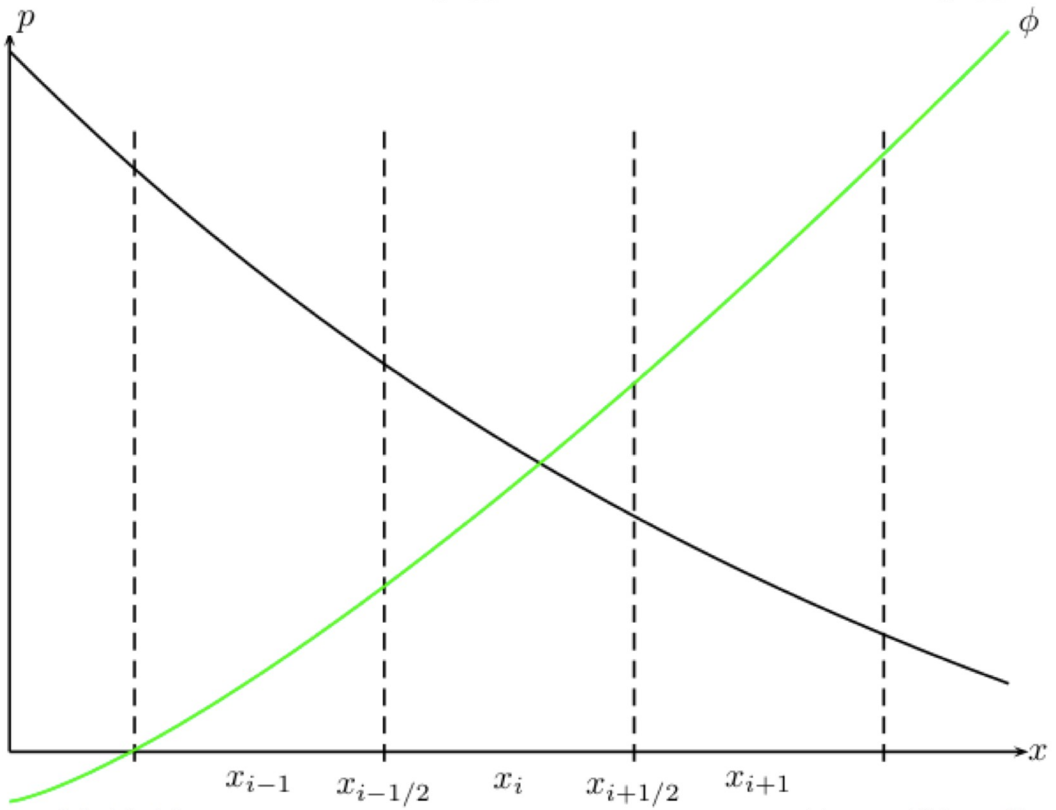

Discretise in cells $[x_{i-1/2}, x_{i+1/2}]$

Well-balanced scheme for HSE (3)

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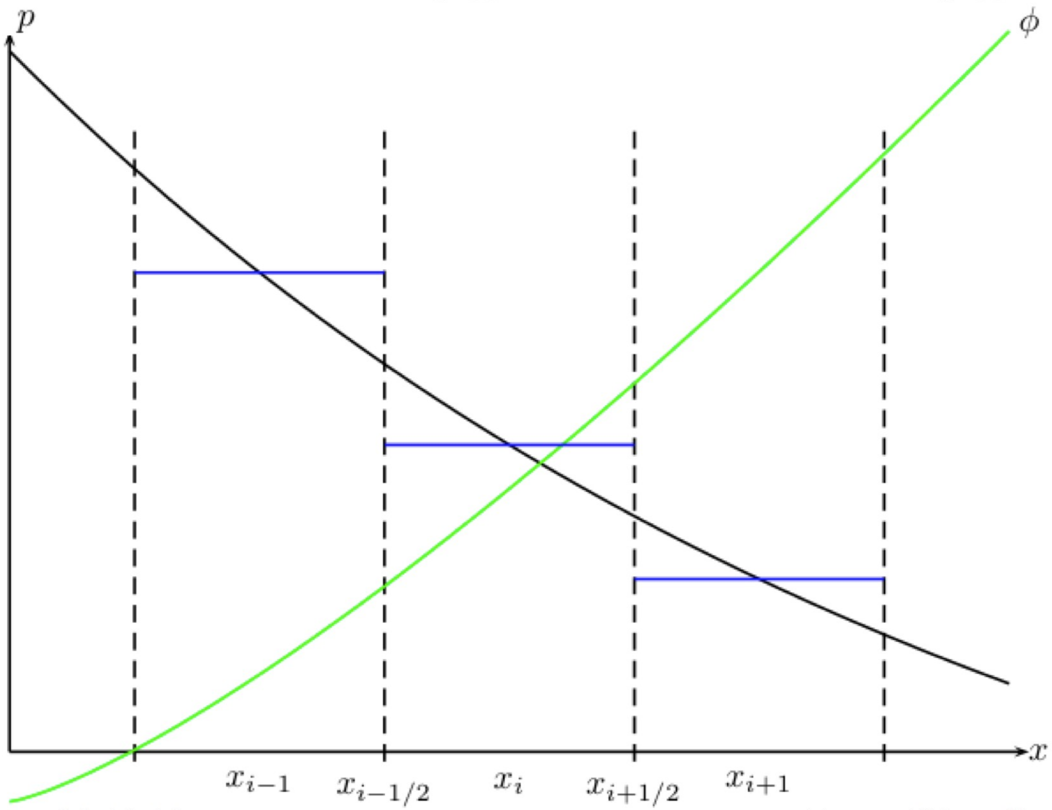


Discretise in cells $[x_{i-1/2}, x_{i+1/2}]$

Define cell averages

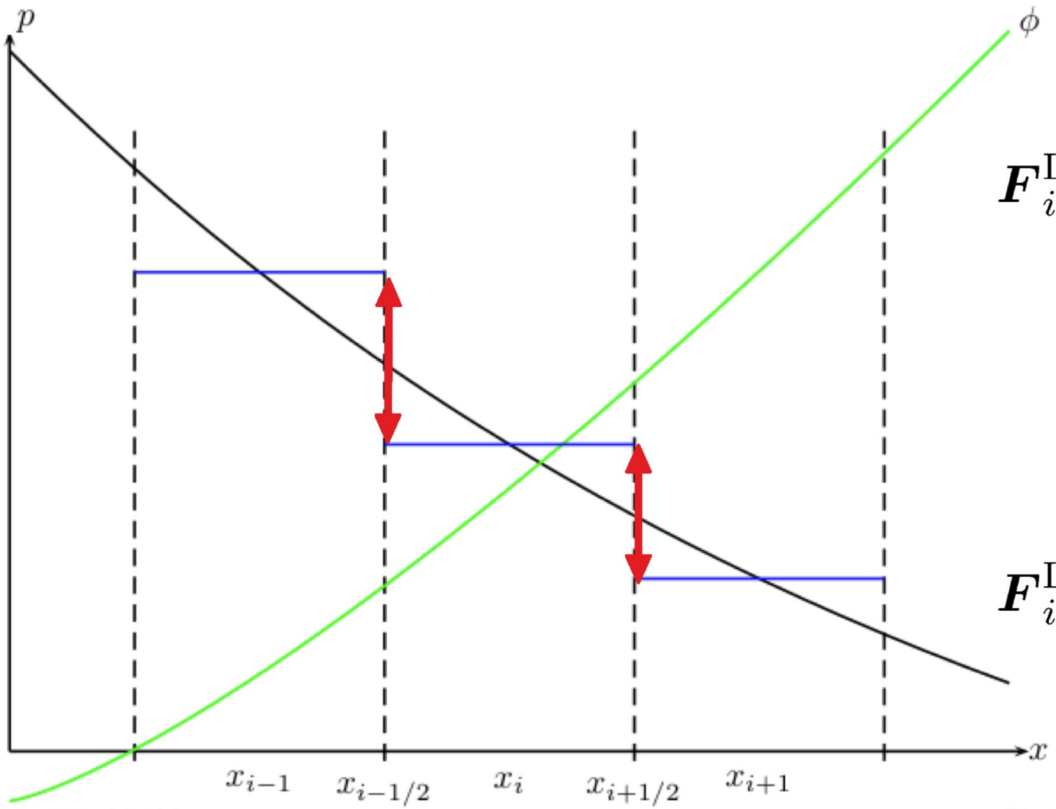
$$\boldsymbol{u}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}(x, t^n) \mathrm{d}x$$

$$\boldsymbol{S}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{S}\left(\boldsymbol{u}(x,t)\right) \mathrm{d}x$$

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) \stackrel{?}{=} \boldsymbol{S}^n_i$$



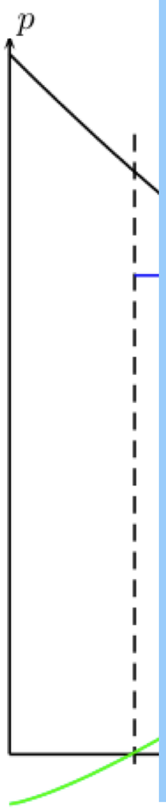
$$\boldsymbol{F}^{\mathrm{LxF}}_{i+1/2} = \frac{1}{2}\left(\boldsymbol{F}_i + \boldsymbol{F}_{i+1}\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i\right)$$

Contains also gravity induced gradient!

$$\boldsymbol{F}^{\mathrm{LxF}}_{i-1/2} = \frac{1}{2}\left(\boldsymbol{F}_{i-1} + \boldsymbol{F}_i\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_i - \boldsymbol{u}_{i-1}\right)$$

Well-balanced scheme for HSE (3)

Intere
equil



Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \qquad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \qquad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0, 2]$$

Error in pressure:

(after 2 sound crossing times)

N	1st	2ndTVD
128	2.1E-02	6.5E-05
256	1.1E-02	1.6E-05
512	5.3E-03	4.1E-06
1024	2.6E-03	1.0E-06
2048	1.3E-03	2.6E-07

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

HLLC numerical flux

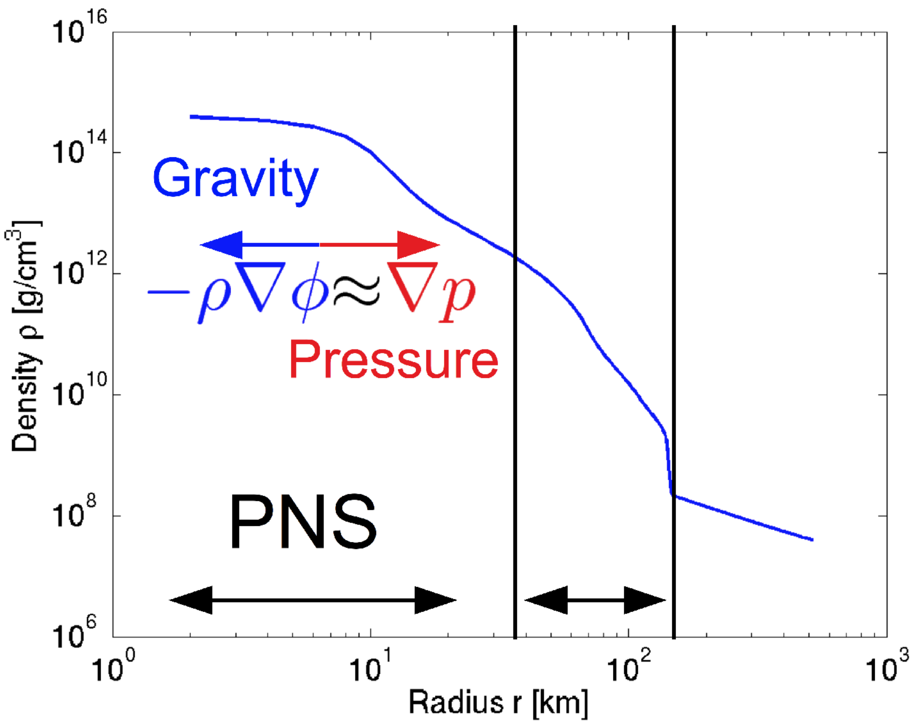
$_{1} - \boldsymbol{u}_i)$

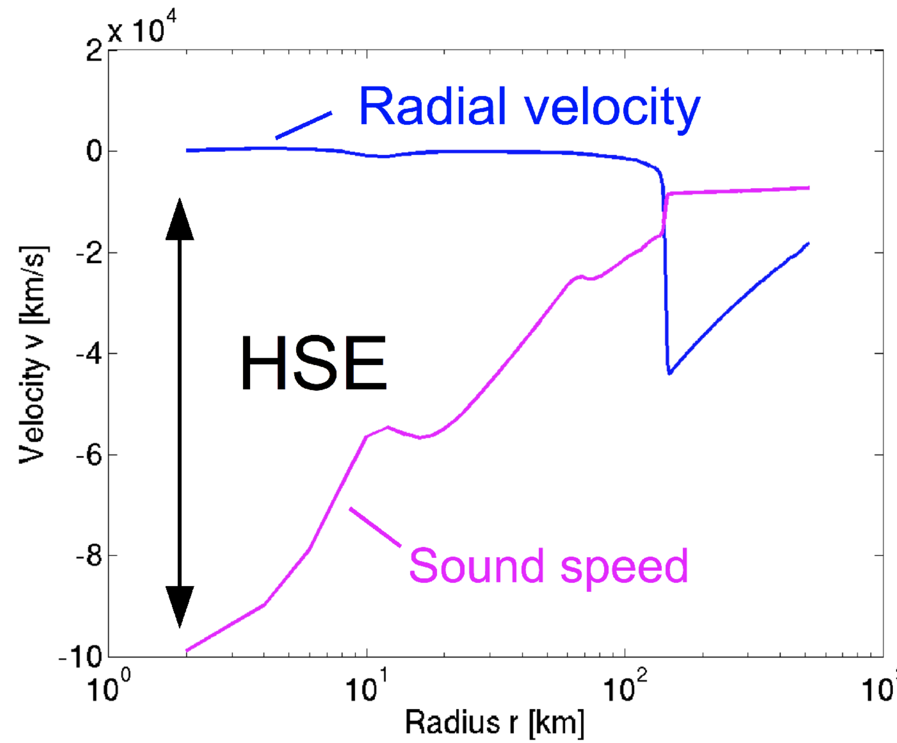
avity
!

$- \boldsymbol{u}_{i-1})$

Well-balanced scheme for HSE (4)

- The problem: (in our simulations)





Ability to maintain near hydrostatic equilibrium for a long time!

Well-balanced scheme for HSE (5)

- Solutions:
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(r)$
  - Steady state preserving reconstructions, well-balanced schemes e.g. LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010), Xing & Shu (2013), Vides et al. (2013)

Note: there are many, many more... especially for shallow-water eqs!!!

Well-balanced scheme for HSE (5)

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Requirements

- Equilibrium not known in advance (self-gravity)
- Extensible for general EoS
- (At least) second order accuracy
- Preserve robustness of base shock capturing scheme

Well-balanced scheme for HSE (6)

- Hydrostatic equilibrium

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

Describes only a mechanical equilibrium...
Density and pressure not uniquely determined

$$p = p(\rho, s) = p(\rho, T)$$

s Entropy

T Temperature

Arbitrary entropy or temperature profiles not (physically) stable (convection!)

Well-balanced scheme for HSE (7)

- Consider constant entropy profile

- Using the thermodynamic relation

$$\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}p}{\rho} \qquad h = e + \frac{p}{\rho}$$ Enthalpy

- Hydrostatic eq.

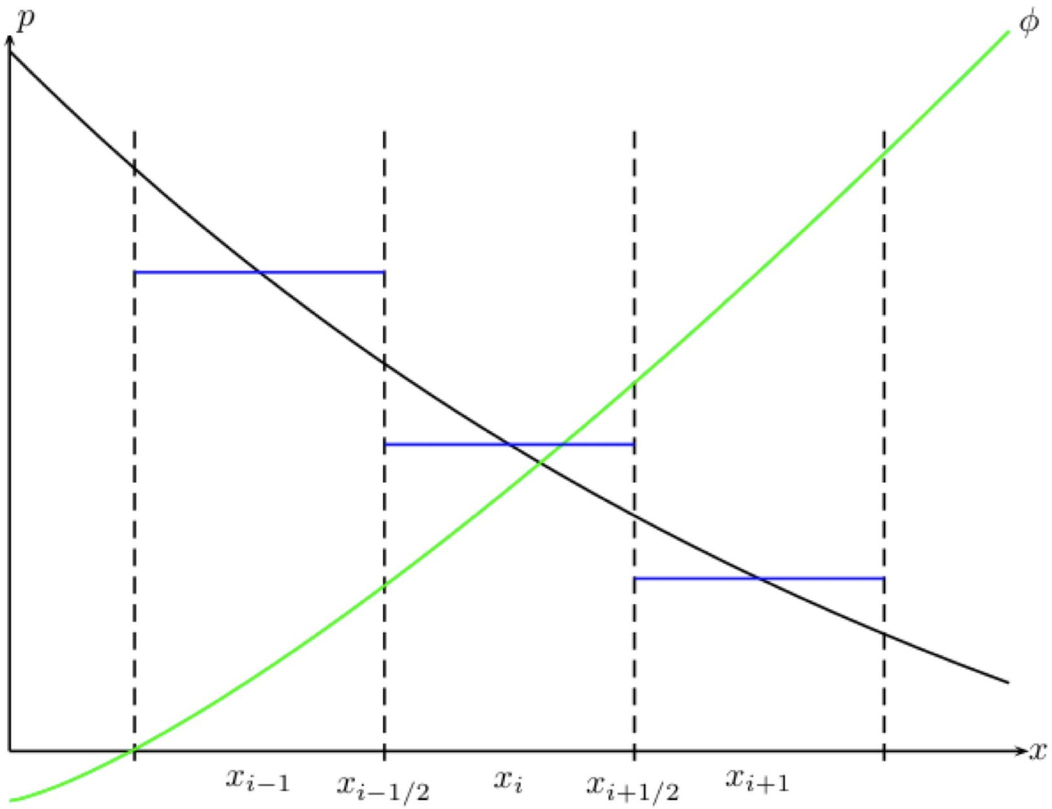
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial h}{\partial x} = -\frac{\partial \phi}{\partial x}$$

- Or simply $h + \phi = const$

Well-balanced scheme for HSE (8)

Perform equilibrium reconstruction:

$$h + \phi = const$$



$$h_{0,i}(x) = h_i + \phi_i - \phi(x)$$

Equilibrium enthalpy

EoS $\downarrow$ $h_{0,i}(x) = h\left(s_i, p_{0,i}(x)\right)$

$p_{0,i}(x)$ & $\rho_{0,i}(x)$

$\downarrow$

$$\boldsymbol{w}^n_{i\pm1/2\mp} = \begin{bmatrix} \rho^n_{0,i}(x_{i\pm1/2}) \\ v^n_{x,i} \\ p^n_{0,i}(x_{i\pm1/2}) \end{bmatrix}$$

Eq. reconstructed primitive variables

Well-balanced scheme for HSE (9)

- Well-balanced discretization of momentum source term

$$S_{\rho v,i}^{n} = \frac{p_{0,i}^{n}(x_{i+1/2}) - p_{0,i}^{n}(x_{i-1/2})}{\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + O(\Delta x^2)$$

- Then for data satisfying $h + \phi = const$, $v_x = 0$
  and any consistent numerical flux

$$\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^{n} - \boldsymbol{F}_{i-1/2}^{n}\right) = \boldsymbol{S}_{i}^{n}$$

Well-balanced wrt hydrostatic equilibrium!

Well-balanced scheme for HSE (10)

- Second order extension: $r_{1,i}(x_j) = r_j - r_{0,i}(x_j)$

$r =$ pressure, density

Eq. perturbation — Data — Equilibrium

Stencil: $j = ..., i-1, i, i+1, ...$

$$r_{1,i}(x) = r_{1,i}(x_i) + Dr_{1,i}(x - x_i) = Dr_{1,i}(x - x_i)$$

$$Dr_{1,i} = \text{limiter}\left(\frac{r_{0,i}(x_{i-1}) - r_{i-1}}{\Delta x}, \frac{r_{i+1} - r_{0,i}(x_{i+1})}{\Delta x}\right)$$

Reconstruction in deviation from equilibrium

Similar to Botta et al. 2004,Fuchs et al. 2010

- Time stepping:

$$\boldsymbol{u}^{*} = \boldsymbol{u}^{n} + \Delta t^{n} \boldsymbol{L}(\boldsymbol{u}^{n})$$

$$\boldsymbol{u}^{**} = \boldsymbol{u}^{*} + \Delta t^{n} \boldsymbol{L}(\boldsymbol{u}^{*})$$

$$\boldsymbol{u}^{n+1} = \frac{1}{2}(\boldsymbol{u}^{n} + \boldsymbol{u}^{**})$$

Strong Stability Preserving Runge-Kutta,
Gottlieb et al. 2001

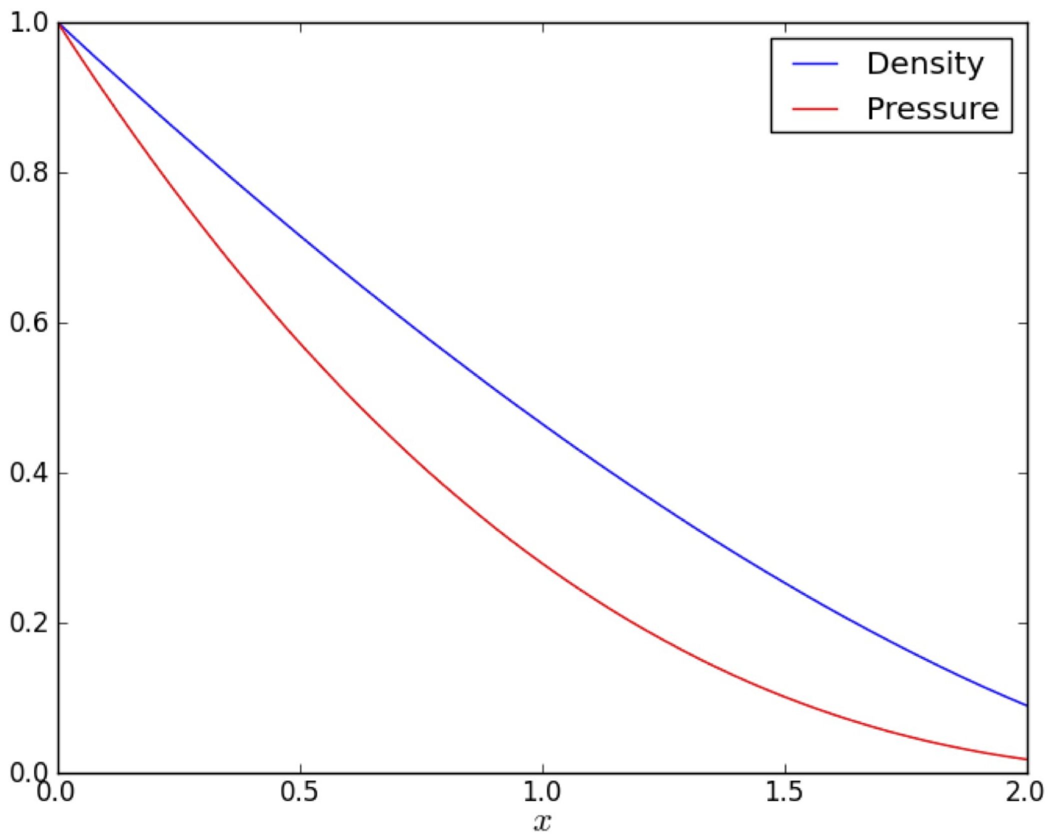
Example 1

Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \quad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \quad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0, 2]$$

$$h + \phi = const$$



Error in pressure:

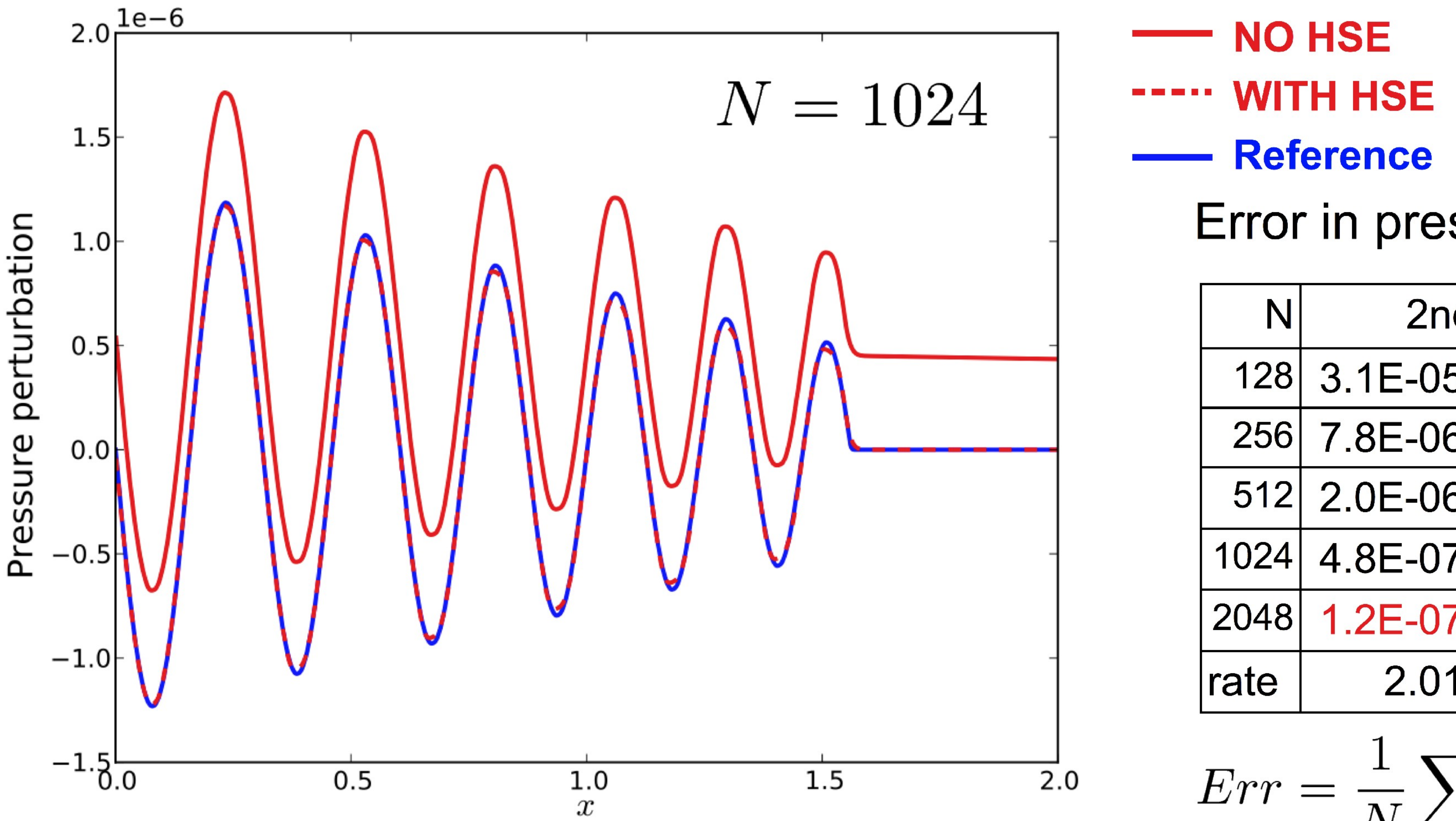
N	1st	2ndTVD
128	2.1E-02 / **1.3E-14**	6.5E-05 / **1.3E-14**
256	1.1E-02 / **3.6E-14**	1.6E-05 / **1.5E-14**
512	5.3E-03 / **7.7E-14**	4.1E-06 / **4.6E-14**
1024	2.6E-03 / **5.7E-14**	1.0E-06 / **6.1E-14**
2048	1.3E-03 / **1.2E-13**	2.6E-07 / **1.5E-14**
rate	1.00 / -	2.00 / -

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

Example 2

Hydrostatic atmosphere in a constant gravitational field

+ small perturbation $v(t, x = 0) = 10^{-6} \sin(8\pi t)$



Error in pressure:

N	2ndTVD
128	3.1E-05 / **1.9E-07**
256	7.8E-06 / **6.8E-08**
512	2.0E-06 / **2.5E-08**
1024	4.8E-07 / **8.5E-09**
2048	1.2E-07 / **4.1E-09**
rate	2.01 / 1.40

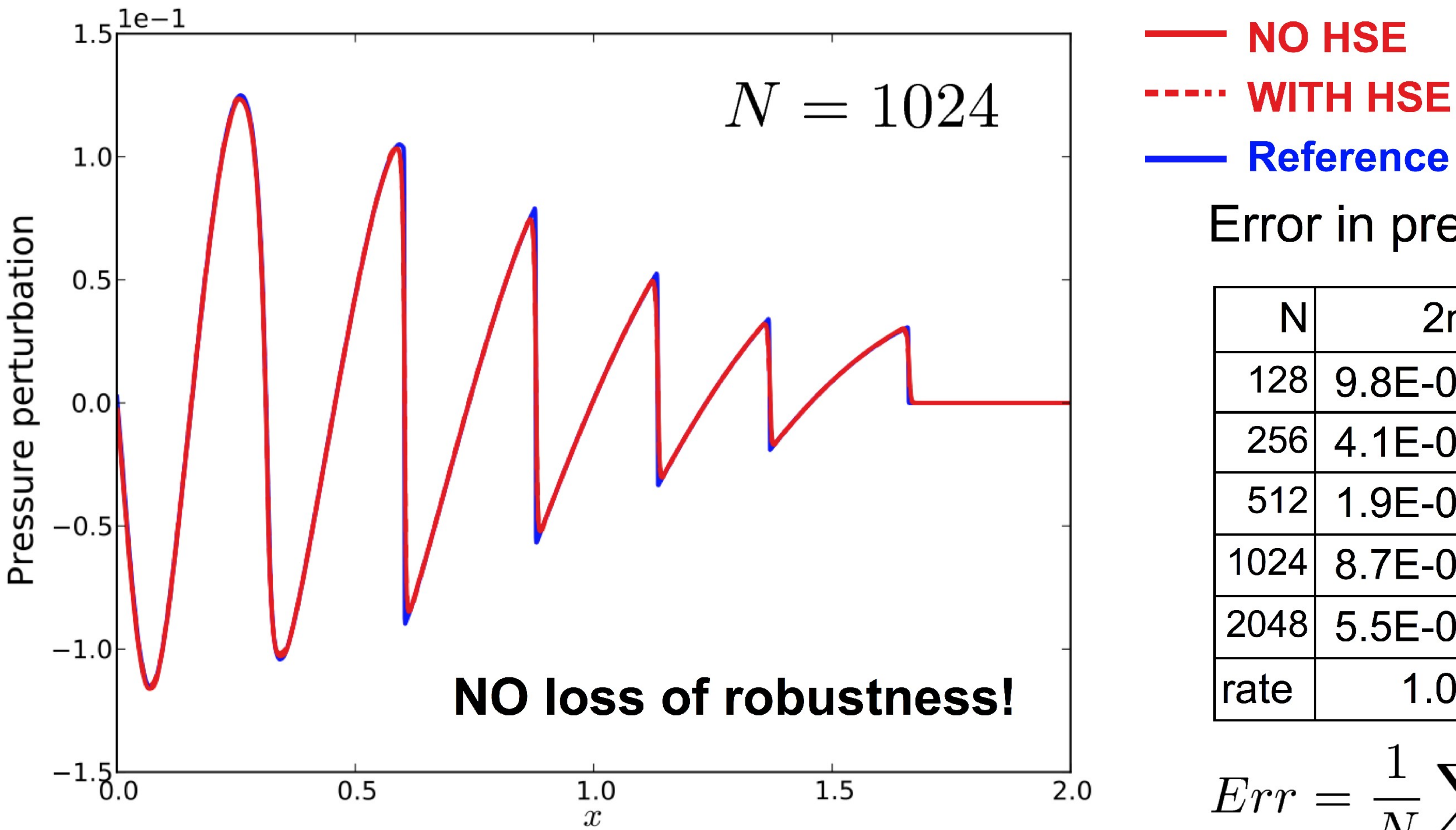
!

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

Example 3

Hydrostatic atmosphere in a constant gravitational field

+ large perturbation $v(t, x=0) = 0.1 \sin(8\pi t)$



Error in pressure:

N	2ndTVD
128	9.8E-03 / **1.1E-02**
256	4.1E-03 / **4.9E-03**
512	1.9E-03 / **2.0E-03**
1024	8.7E-04 / **8.0E-04**
2048	5.5E-04 / **3.3E-04**
rate	1.05 / 1.28

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

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Multi-dimensional extension

- Straight forward directional application of HydroStatic Reconstruction

$$\frac{\mathrm{d}\boldsymbol{u}_{i,j}}{\mathrm{d}t} = \boldsymbol{L}(\boldsymbol{u}) = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2,j} - \boldsymbol{F}_{i-1/2,j}\right) - \frac{1}{\Delta y}\left(\boldsymbol{G}_{i,j+1/2} - \boldsymbol{G}_{i,j-1/2}\right) + \boldsymbol{S}_{i,j}$$

- Hydrostatic equilibrium:

$$h + \phi = const$$

Example 4

Polytrope: model star (e.g. main sequence stars, white dwarfs, neutron stars)

HSE: $\nabla p = -\rho \nabla \phi$ Poisson equation: $\nabla^2 \phi = -4\pi G \rho$

Equation of state $p = K\rho^\gamma$ $K = 1$

Take $\gamma = 2$ ~ neutron stars

Then there's an exact solution:

Central density

$$\rho(\boldsymbol{x}) = \rho_c \frac{\sin(\alpha r)}{r}$$

$$\phi(\boldsymbol{x}) = -\gamma K \rho(\boldsymbol{x})$$

$$\alpha = \sqrt{\frac{2K}{4\pi G}} \quad r = \sqrt{x^2 + y^2 + z^2}$$

Example 4

Po

H

E

Ta

Th

ρ

Evolution for 20 "sound crossing" times

NO HSE

WITH HSE

Reference

Error in density:

N	2ndTVD
32	1.3E-02 / **1.5E-14**
64	3.6E-03 / **3.0E-14**
128	1.0E-03 / **5.6E-14**
rate	1.82 / **-**

Example 4

Po

Small perturbation $\rho(\boldsymbol{x}) = \rho_0(\boldsymbol{x}) + Ae^{-100\boldsymbol{x}^2} \quad A = 10^{-3}$

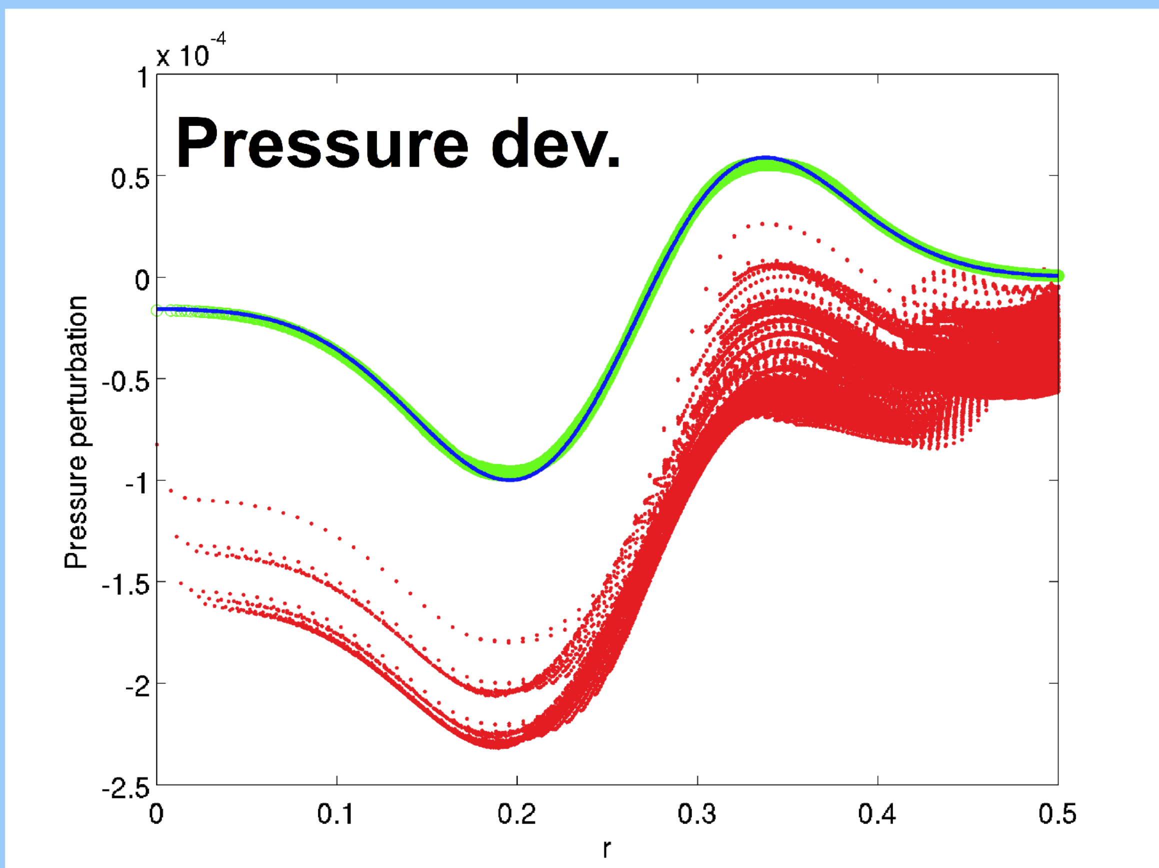
H

ρ

E

Ta

Th



NO HSE

WITH HSE

Reference

Error in pressure:

N	2ndTVD
32	8.4E-04 / **1.1E-06**
64	2.1E-04 / **3.7E-07**
128	5.1E-05 / **1.1E-07**
rate	2.02 / **1.67**

!

Example 4

Pc

H

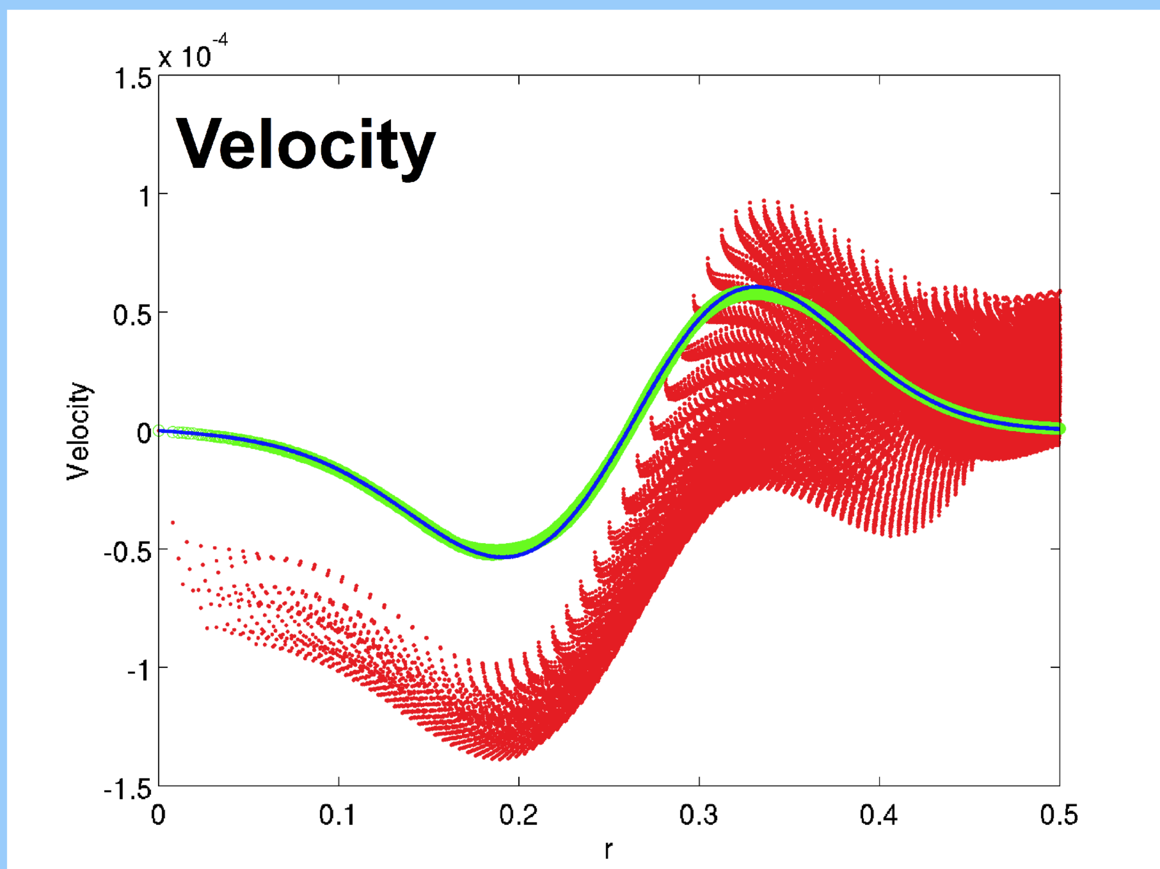
Ec

Ta

Th

ρ

Small perturbation $\rho(\boldsymbol{x}) = \rho_0(\boldsymbol{x}) + Ae^{-100\boldsymbol{x}^2}$ $A = 10^{-3}$



NO HSE

WITH HSE

Reference

Error in velocity:

N	2ndTVD
32	6.0E-04 / **8.8E-07**
64	1.6E-04 / **3.0E-07**
128	4.1E-05 / **8.5E-08**
rate	1.92 / **1.69**

!

Example 4

Po

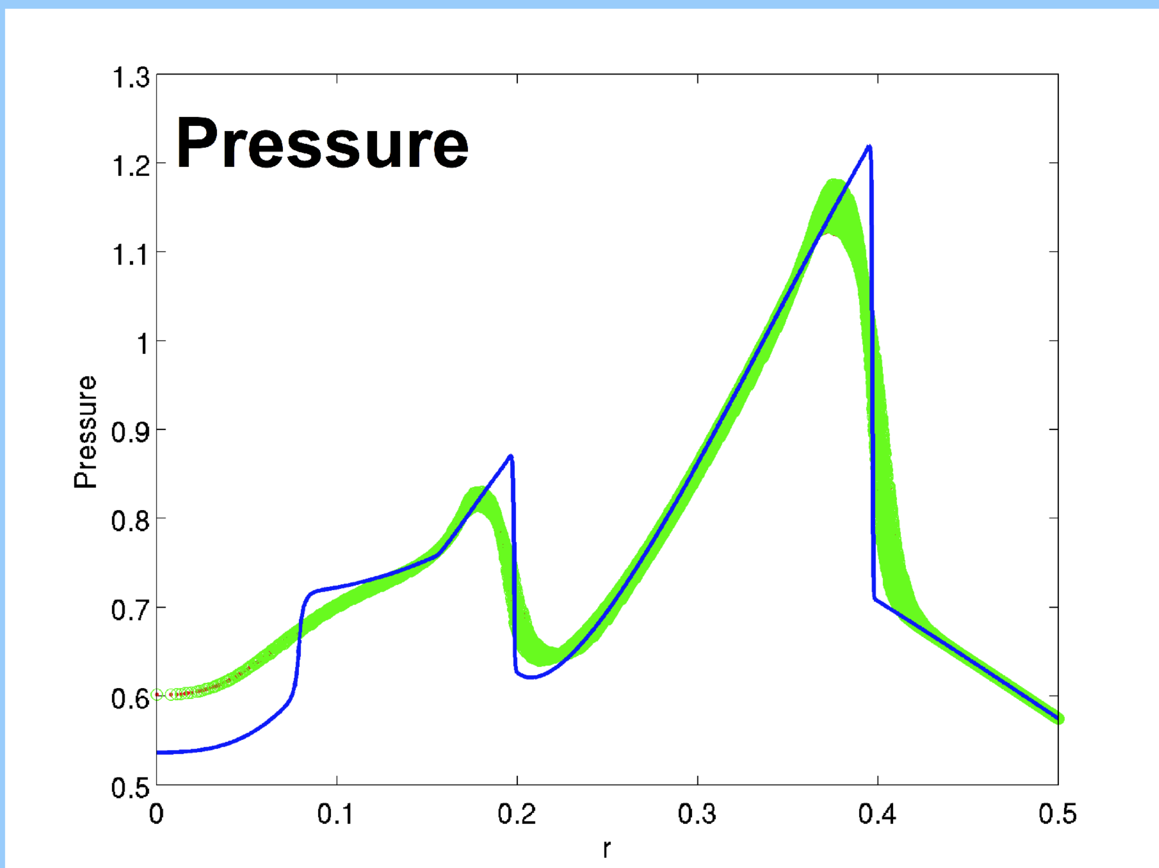
H ρ

E

Ta

Th

Large perturbation (detonation!)



NO HSE
WITH HSE
Reference

Error in pressure:

N	2ndTVD
32	3.3E-02 / **3.3E-02**
64	1.8E-02 / **1.8E-02**
128	9.6E-03 / **9.5E-03**
rate	0.90 / **0.89**

Example 4

Po

H

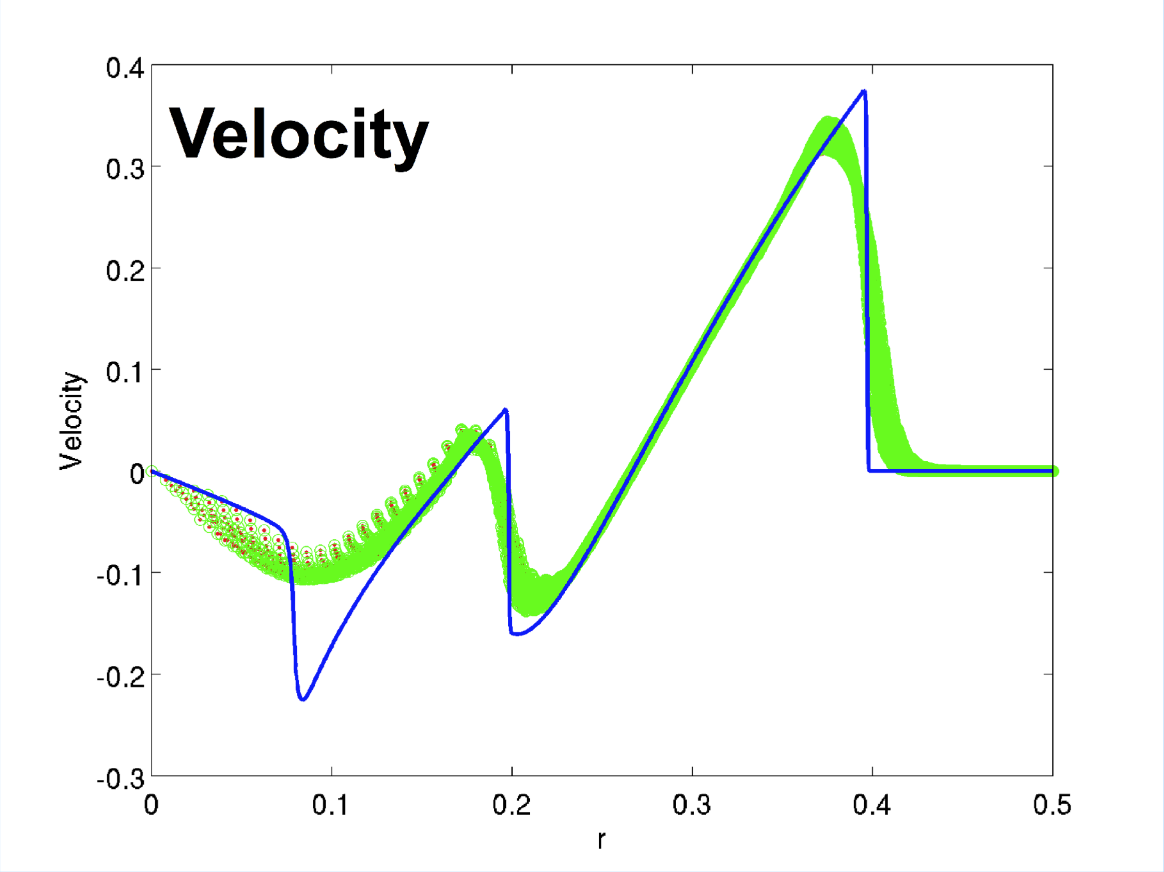
E

Ta

Th

ρ

Large perturbation (detonation!)



Error in velocity:

N	2ndTVD
32	2.9E-02 / **2.8E-02**
64	1.5E-02 / **1.4E-02**
128	7.6E-03 / **7.6E-03**
rate	0.96 / **0.94**

Example 4

Po

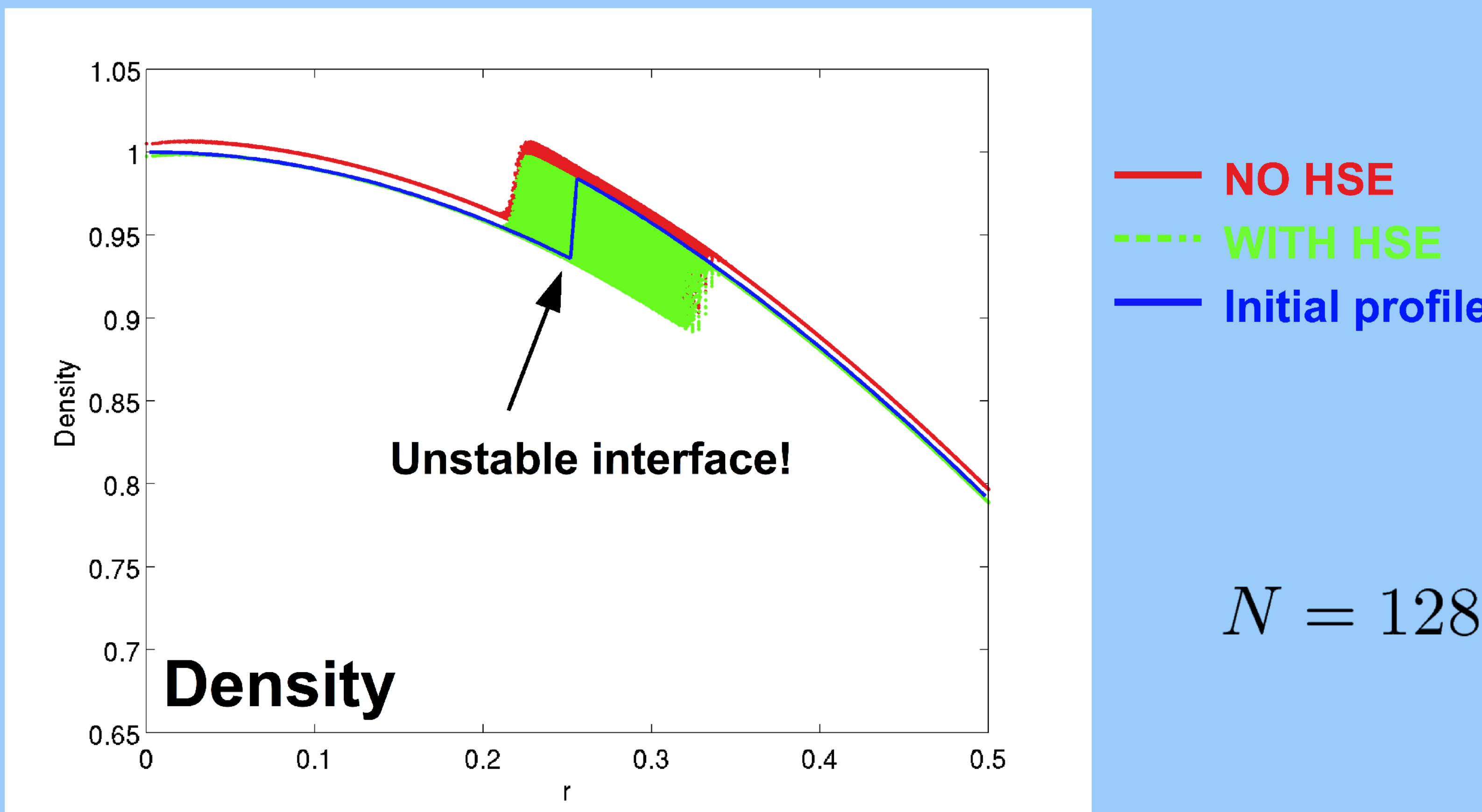
H ρ

E

Ta

Th

Rayleigh-Taylor instability



$$N = 128$$

Example 4

Po

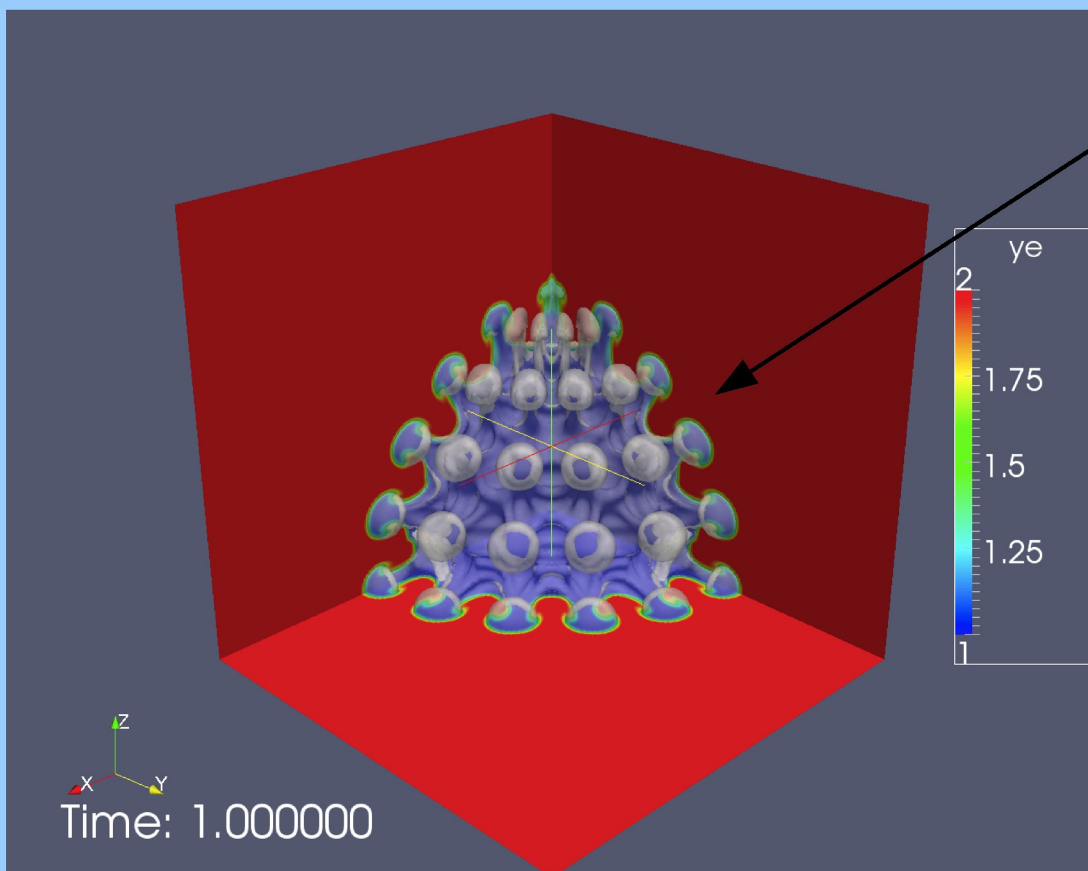
H ρ

Ec

Ta

Th

Rayleigh-Taylor instability



Rayleigh-Taylor “mushrooms”

$$N = 128$$

Well-balanced scheme for HSE

- Consider constant temperature profile

- Using the thermodynamic relation

$$\mathrm{d}g = -s\mathrm{d}T + \frac{\mathrm{d}p}{\rho} \qquad g = h - Ts$$

Gibbs free energy

- Hydrostatic eq.

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial g}{\partial x} = -\frac{\partial \phi}{\partial x}$$

- Or simply $g + \phi = const$ → Reconstruction...

Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) = \boldsymbol{S}^n_i$$

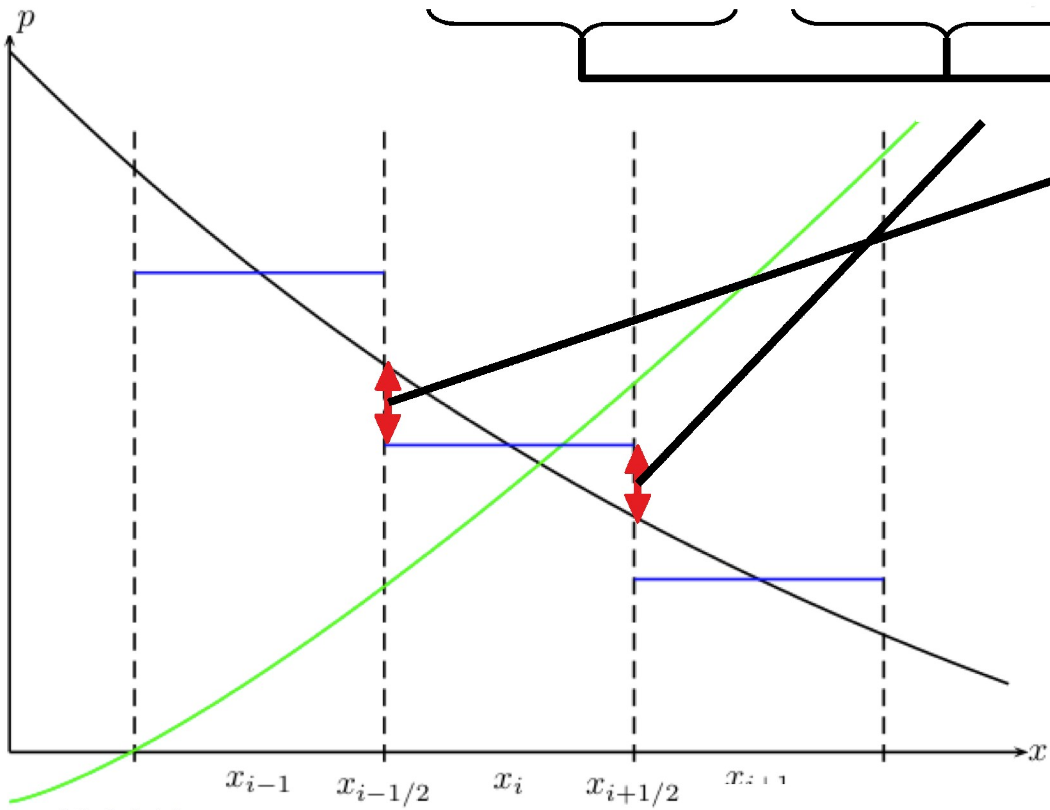
$$\frac{\partial p}{\partial x} + O(\Delta x^2) = \frac{p_{i+1/2} - p_{i-1/2}}{\Delta x} = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)$$

$$\frac{(p_{i+1/2} - p_i) - (p_{i-1/2} - p_i)}{\Delta x} = -\frac{\rho_i}{2} \frac{(\phi_{i+1} - \phi_i) - (\phi_{i-1} - \phi_i)}{\Delta x}$$

Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

$$\frac{p_{i+1/2}-p_i}{\Delta x}-\frac{p_{i-1/2}-p_i}{\Delta x}=-\frac{\rho_i}{2}\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}-\frac{\phi_{i-1}-\phi_i}{\Delta x}\right)$$



Equilibrium reconstruction:

$$p_{i+1/2}=p_i+\frac{\Delta x}{2}\Delta p_i^+$$

$$p_{i-1/2}=p_i-\frac{\Delta x}{2}\Delta p_i^-$$

Equilibrium differences:

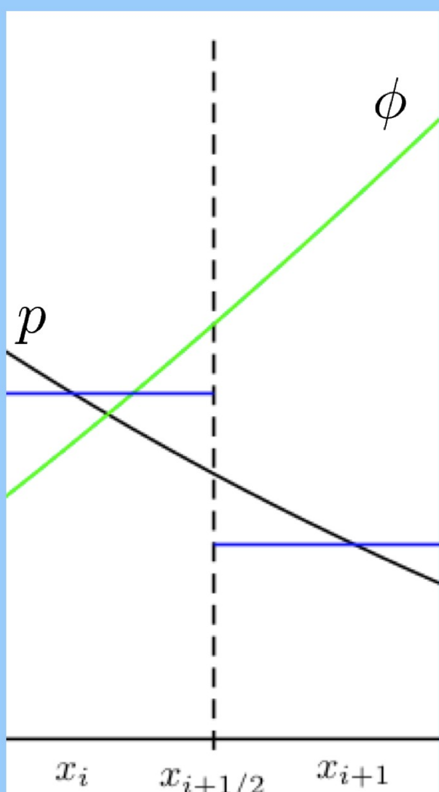
$$\Delta p_i^+=-\rho_i\frac{\phi_{i+1}-\phi_i}{\Delta x}$$

$$\Delta p_i^-=-\rho_i\frac{\phi_i-\phi_{i-1}}{\Delta x}$$

Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

Equilibrium?



$$p^L_{i+1/2} \overset{!}{=} p^R_{i+1/2}$$

$$p_i + \frac{\Delta x}{2}\Delta p_i^+ = p_{i+1} - \frac{\Delta x}{2}\Delta p_{i+1}^-$$

$$\frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2}\frac{\phi_{i+1} - \phi_i}{\Delta x}$$

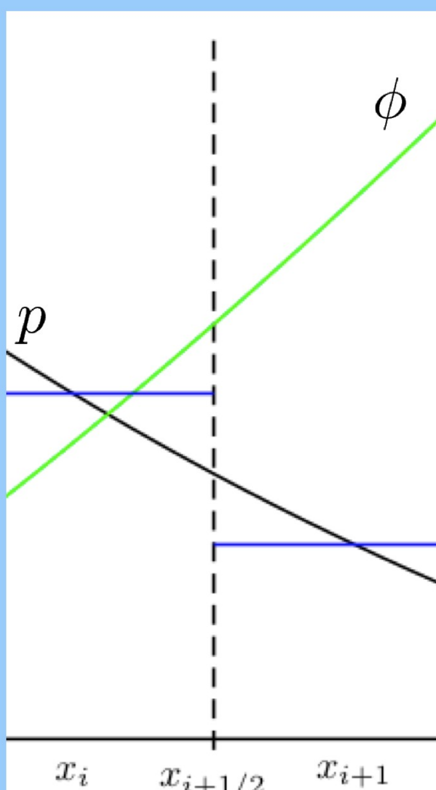
Discrete HydroStatic Equilibrium

28.

Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

Equilibrium?



$$p^L_{i+1/2} \overset{!}{=} p^R_{i+1/2}$$

Requirement on Riemann solver:

$$\boldsymbol{F}^n_{i\pm 1/2} = \boldsymbol{\mathcal{F}}\left(\begin{bmatrix} \rho^L_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix}, \begin{bmatrix} \rho^R_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ p_{i+1/2} \\ 0 \end{bmatrix}$$

e.g. HLLC, Roe

Discrete HydroStatic Equilibrium

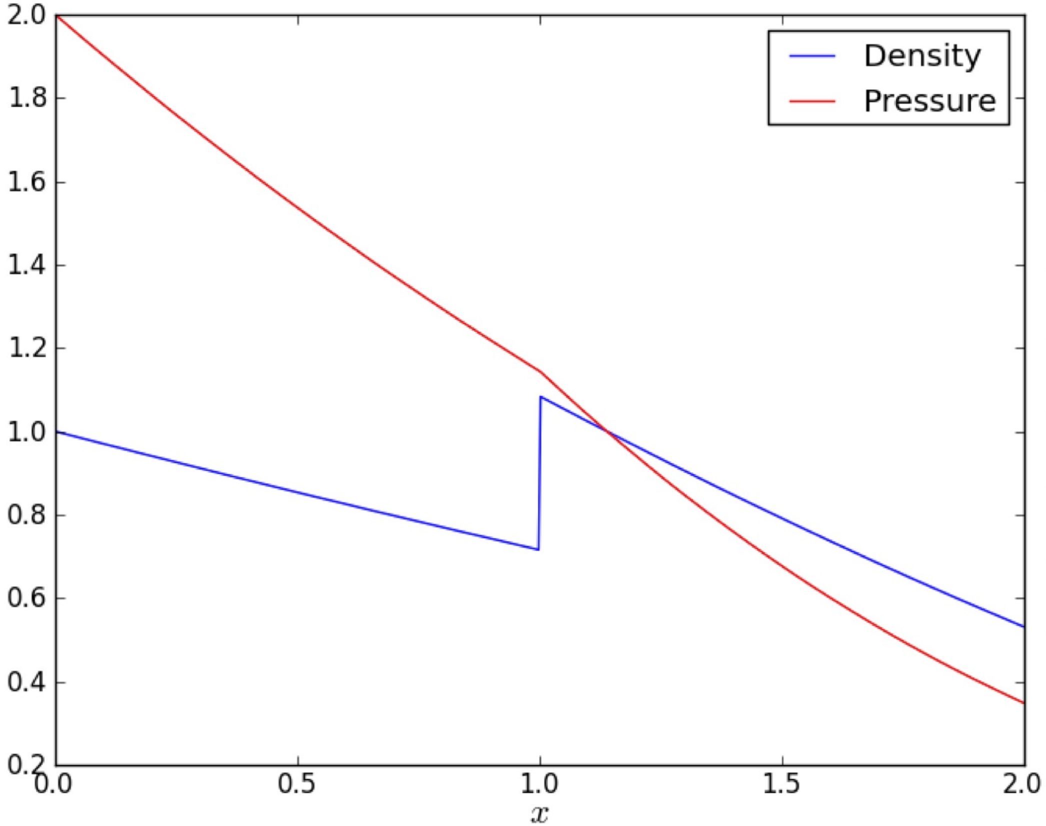
Example 5

Hydrostatic atmosphere in a constant gravitational field

$$\phi_i = g x_i \qquad \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} \qquad p_i = K_i \rho_i^\gamma$$

$$x \in [0, 2]$$

$$K = \begin{cases} 2 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$ ~ entropy



Error in pressure:

N	1st	2ndTVD
32	6.3E-02 / **3.3E-16**	6.2E-04 / **1.3E-16**
64	3.2E-02 / **3.8E-15**	1.6E-04 / **4.6E-16**
128	1.6E-02 / **6.1E-15**	4.2E-05 / **8.8E-16**
256	8.0E-03 / **7.0E-15**	1.1E-05 / **6.7E-16**
512	4.0E-03 / **1.1E-13**	2.7E-06 / **3.4E-15**

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

Conclusions

- 1D well-balanced scheme for (isentropic, isothermal, arbitrary) hydrostatic equilibrium (for general EoS)
- Extension to higher-order?
- Non-zero velocity steady state?

  (e.g. for steady accretion flow)
- Multi-D well-balanced scheme for (isentropic, isothermal) hydrostatic equilibrium (for general EoS)

Thank you for your attention!!!