Using the Jacobian-free Newton-Krylov method to solve the sea-ice momentum equation

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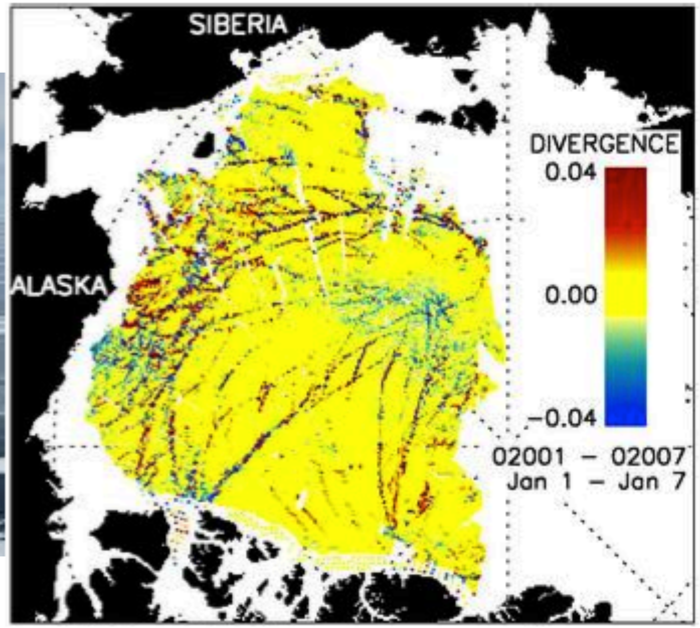
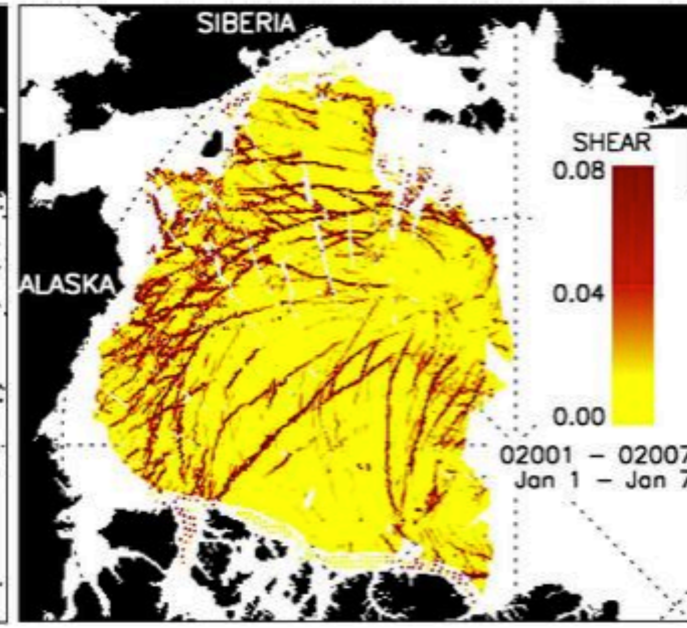
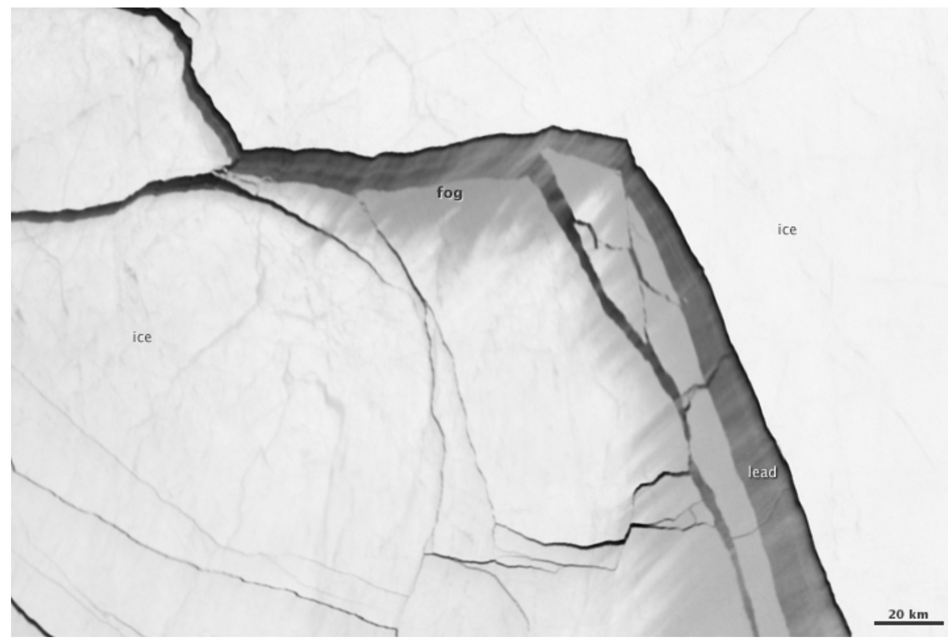

SIBERIA
ALASKA
DIVERGENCE
0.04
0.00
−0.04
02001 – 02007
Jan 1 – Jan 7


SIBERIA
ALASKA
SHEAR
0.08
0.04
0.00
02001 – 02007
Jan 1 – Jan 7


fog
ice
ice
lead
20 km

The sea-ice momentum equation

Assumption: Long enough time step

$$-\rho_i h f \mathbf{k} \times \mathbf{u}_i + a(\tau_a - \tau_w) + \nabla \cdot \sigma - \rho_i h g \nabla H_d = 0$$

The sea-ice momentum equation

Assumption: Long enough time step

Coriolis force

rheology

$$-\rho_i h f \mathbf{k} \times \mathbf{u}_i + a(\tau_a - \tau_w) + \nabla \cdot \sigma - \rho_i h g \nabla H_d = 0$$

wind stress

water drag

sea surface tilt

The sea-ice momentum equation

Note: wind speed generally much larger than sea-ice speed

non-linear terms

rheology

$$-\rho_i h f \mathbf{k} \times \mathbf{u}_i + a(\tau_a - \tau_w) + \nabla \cdot \sigma - \rho_i h g \nabla H_d = 0$$

water drag

Introduction to rheology

1-D Example:

stress < critical value ⟶ elastic material ⟶ deformations reversible

stress = critical value ⟶ plastic material ⟶ deformations not reversible

stresses cannot be larger than critical value

Introduction to rheology

1-D Example:

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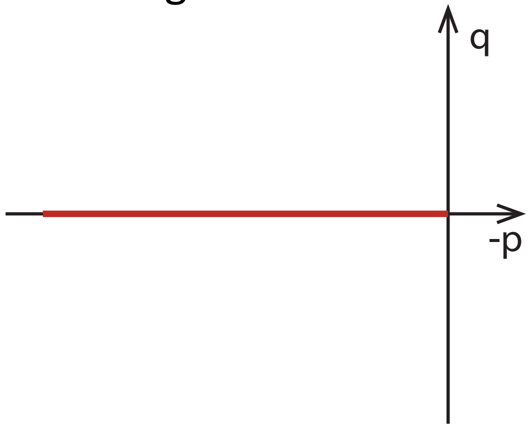
stresses cannot be larger than critical value

BUT: storage and numerically expensive

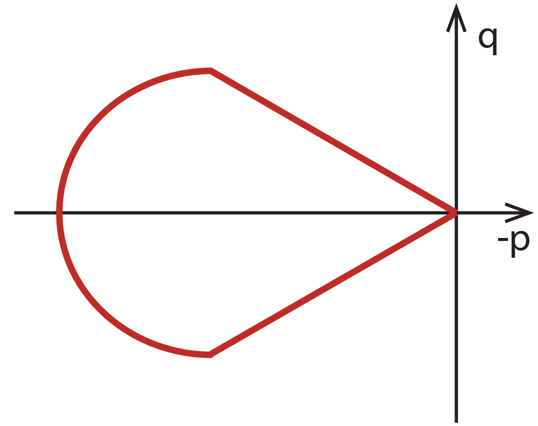
sea ice modeled as viscous-plastic material
(very viscous / creep flow)

Yield curve

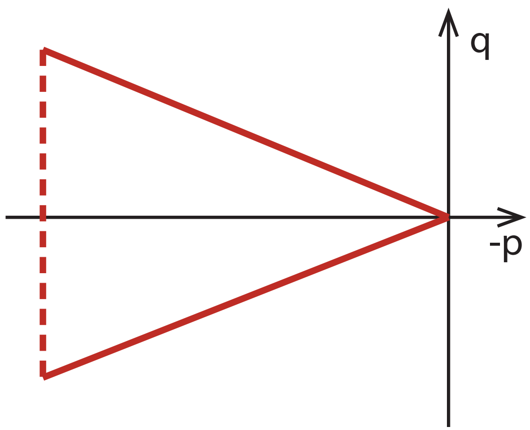
cavitating fluid



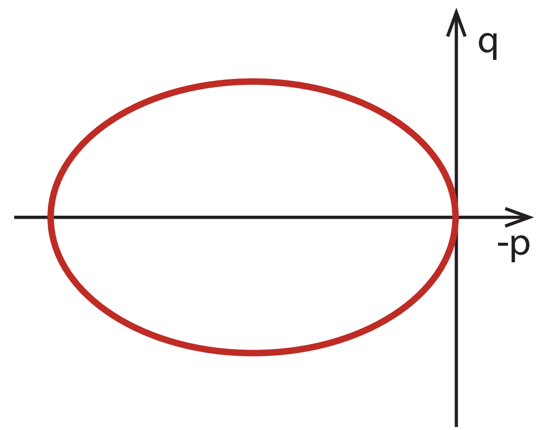
ice cream cone



granular



Hibler model; EVP



Constitutive law (i.e., relation between stress and strain rate) and normal flow rule

$$\sigma_{ij} = 2\eta\dot{\epsilon}_{ij} + [\zeta - \eta]\dot{\epsilon}_{kk}\delta_{ij} - P\delta_{ij}/2$$

Constitutive law (i.e., relation between stress and strain rate) and normal flow rule

ice strength parameter

ice concentration

$$P = P^* h \, exp[-C(1-A)]$$

ice thickness

ice concentration parameter

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Constitutive law (i.e., relation between stress and strain rate) and normal flow rule

strain rates

$$\dot{\epsilon}_{11} = \frac{\partial u}{\partial x} \quad \dot{\epsilon}_{22} = \frac{\partial v}{\partial y}$$

$$\dot{\epsilon}_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\dot{\epsilon}_{kk} = \dot{\epsilon}_{11} + \dot{\epsilon}_{22}$$

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Kronecker Delta

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Kronecker Delta

bulk viscosity

shear viscosity

$$\zeta = \frac{P}{2\Delta}$$

$$\eta = \zeta e^{-2}$$

$$\Delta = [(\dot{\epsilon}_{11}^2 + \dot{\epsilon}_{22}^2)(1 + e^{-2}) + 4e^{-2}\dot{\epsilon}_{12}^2 + 2\dot{\epsilon}_{11}\dot{\epsilon}_{22}(1 - e^{-2})]^{\frac{1}{2}}$$

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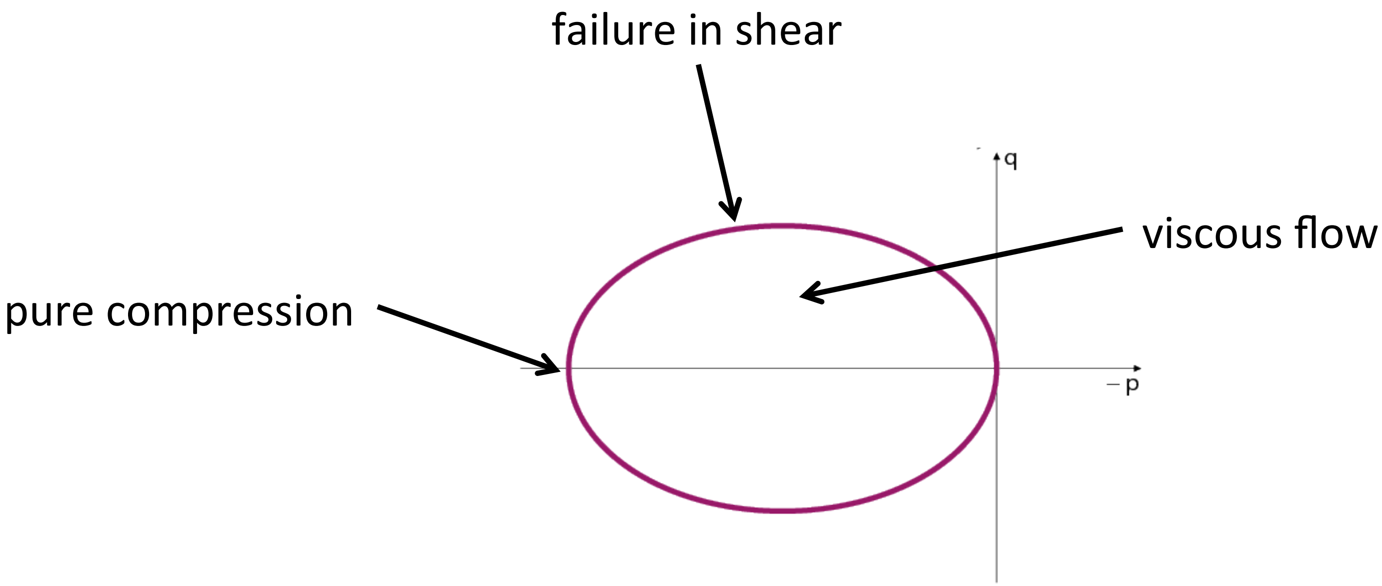
using: $\zeta = min\left(\frac{P}{2\Delta}, \zeta_{max}\right)$ $\qquad \eta = min\left(\frac{P}{2e^2\Delta}, \eta_{max}\right)$

… and now graphically



outside yield curve: not physical

Brief history

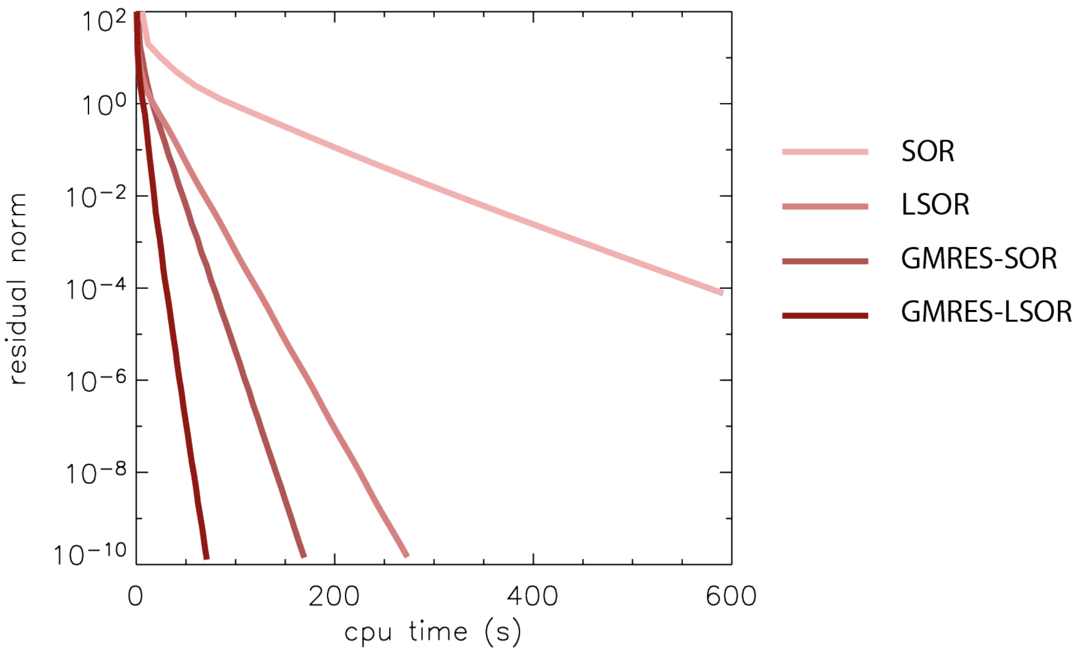
1979: viscous-plastic sea ice model — outer loop; successive over-relaxation, linear relaxation

1992: cavitating sea-ice model — only pressure

1997: elastic-viscous-plastic sea-ice model — artificial elastic term; explicit; conjuate gradient

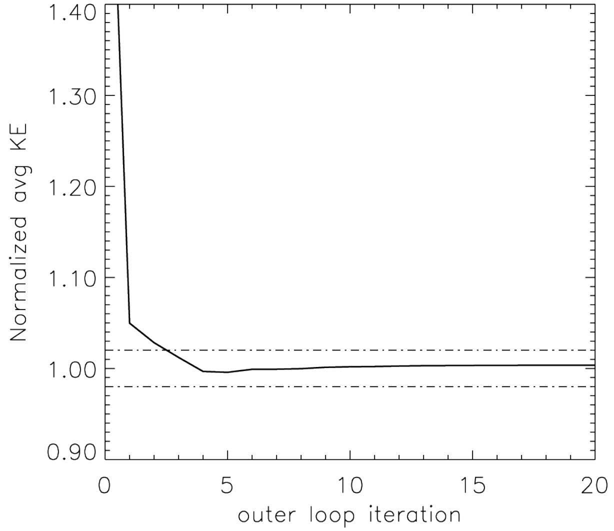
Difference SOR vs. GMRES

only for the linear solver



Outer loop convergence

Kinetic energy



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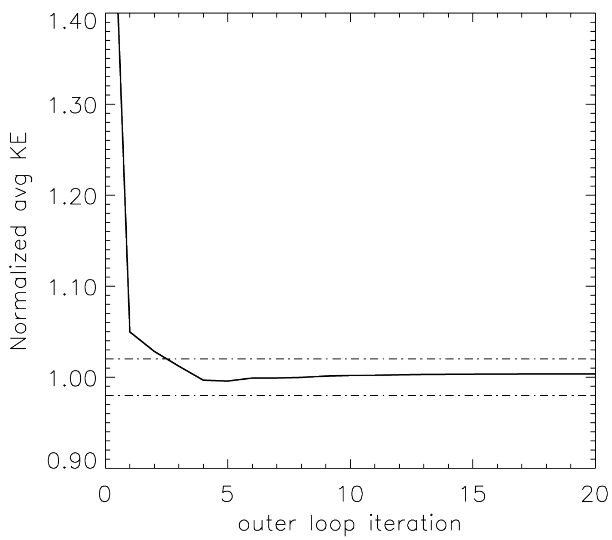
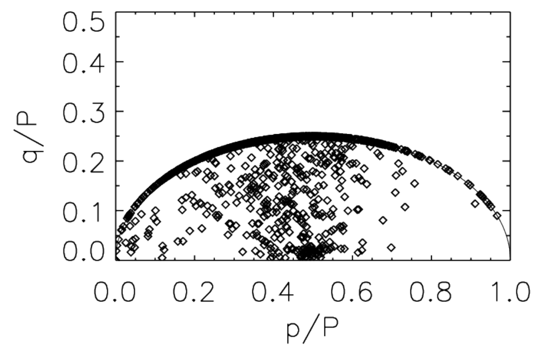
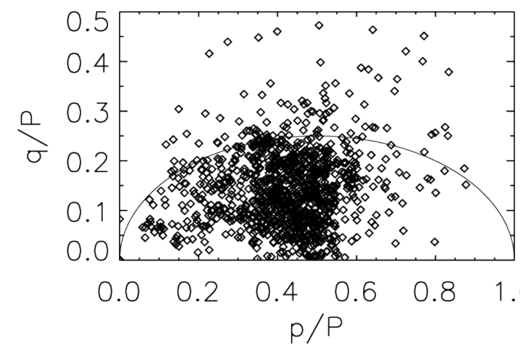
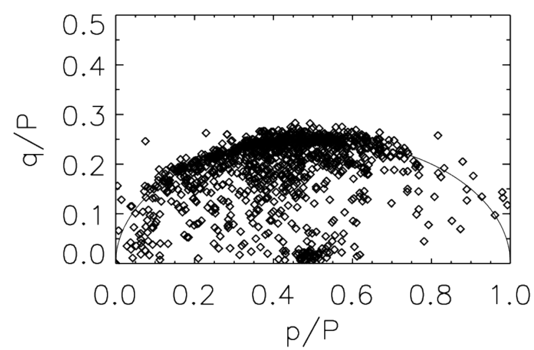


10’500 OL iterations

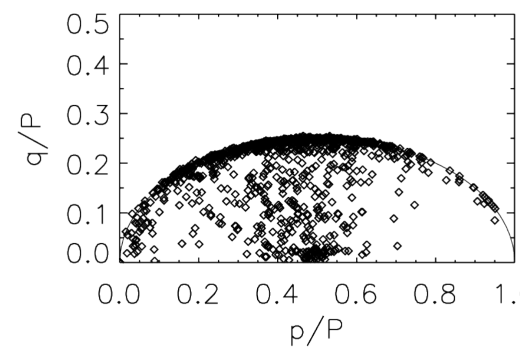


2 OL iterations





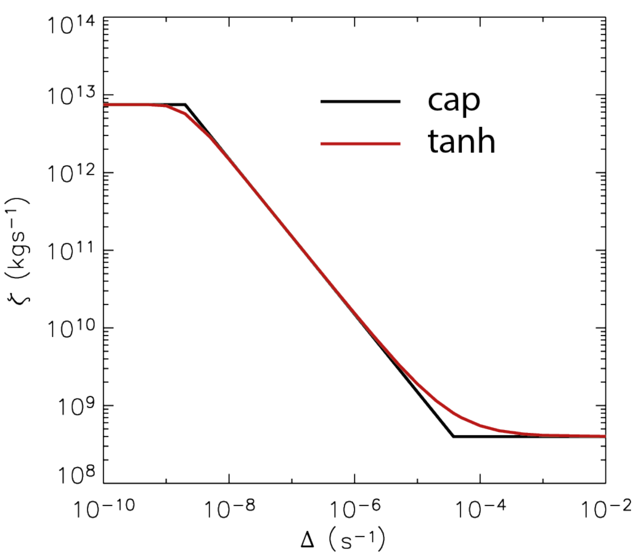
10 OL iterations

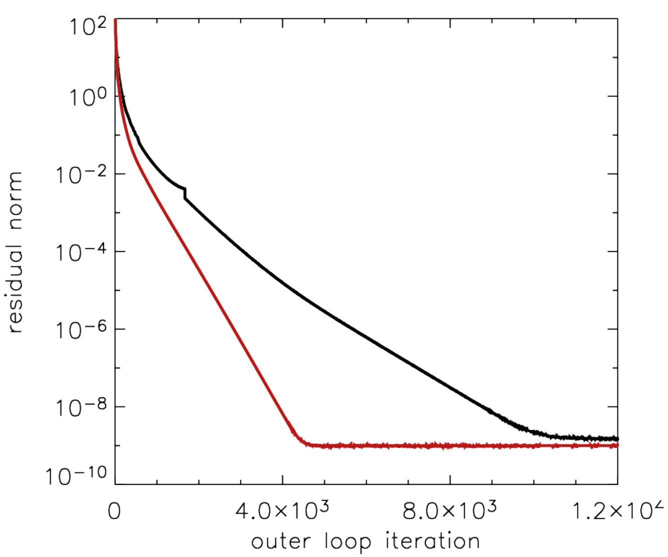


40 OL iterations

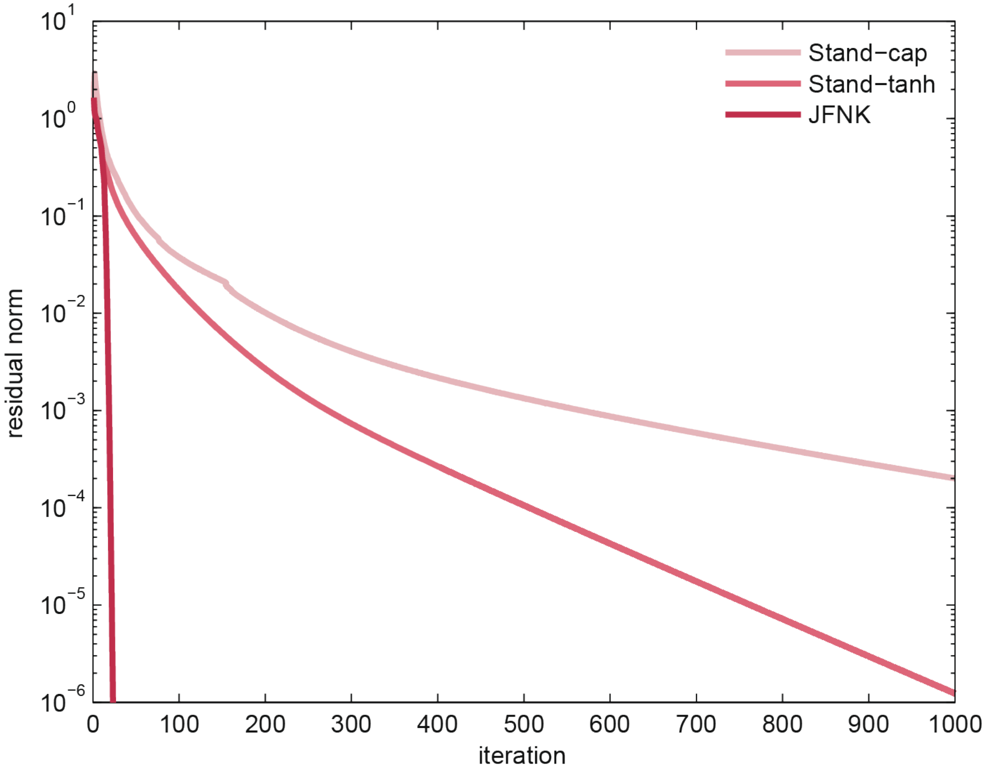
Removing discontinuity

$$\zeta = min\left(\frac{P}{2\Delta}, \zeta_{max}\right) \longrightarrow \zeta = \zeta_{max}\, tanh\left(\frac{P}{2\Delta\zeta_{max}}\right) + \zeta_{min}$$

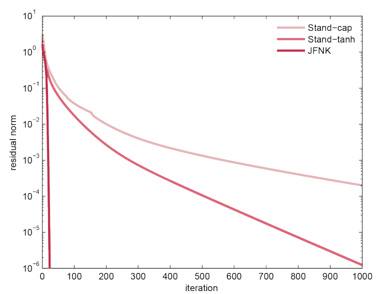


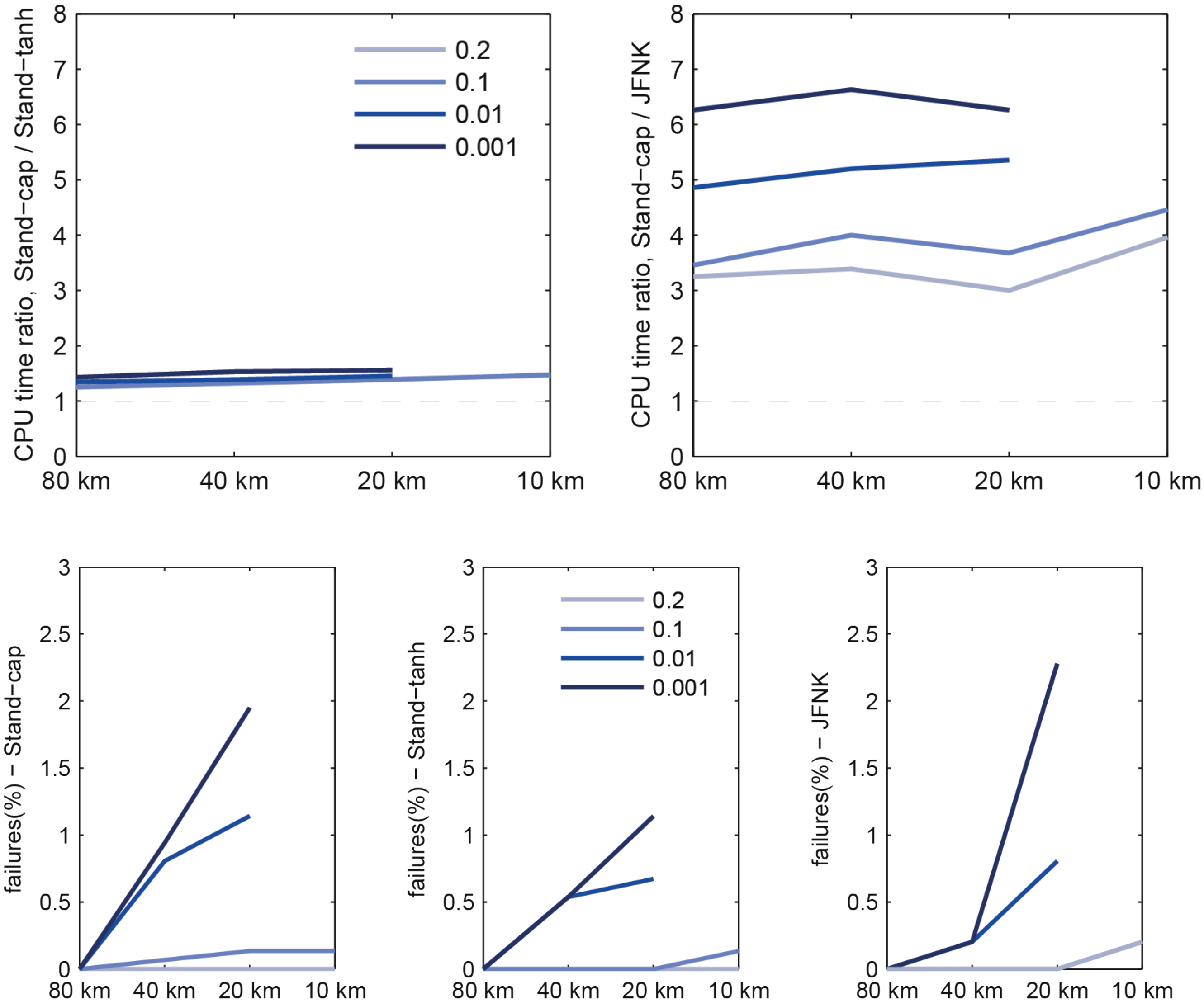


Full Jacobian-free Newton-Krylov



Full Jacobian-free Newton-Krylov

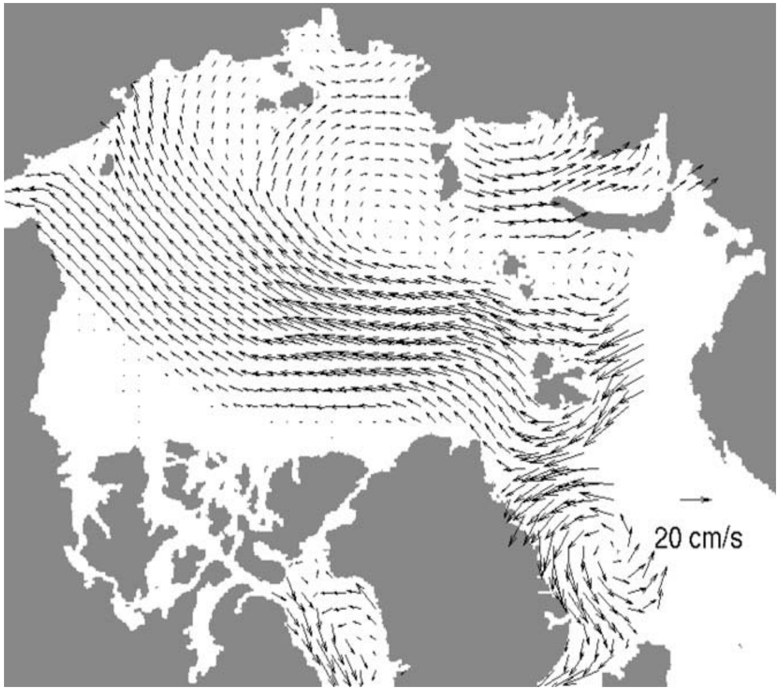




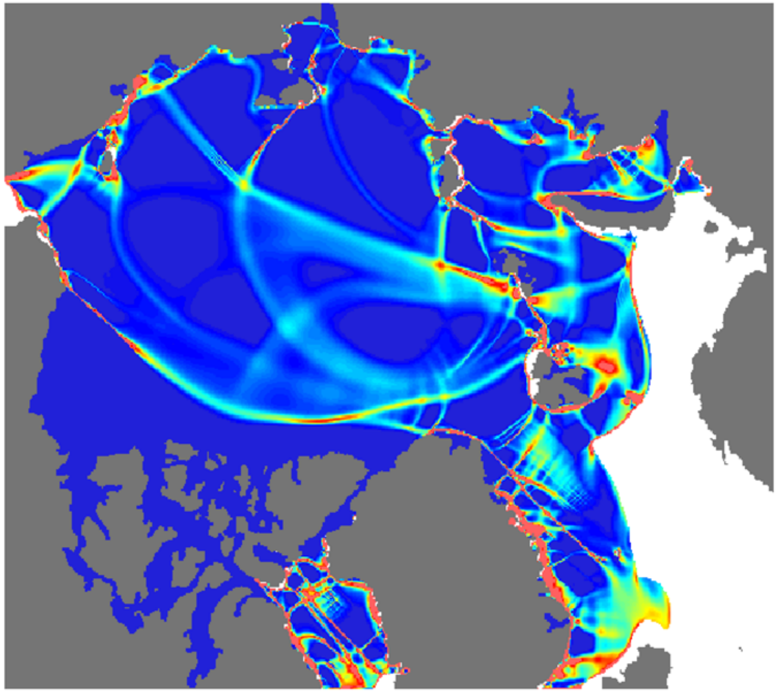
The final result

simulation with 10 km resolution

sea-ice velocity



shear deformation



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