Well-balanced schemes for the Euler equations with gravitation

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Joint work with S. Mishra

Seminar for Applied Mathematics

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Outline

- Introduction
- Well-balanced scheme for HydroStatic Equilibrium (HSE)
  - First order
  - Second order
- Multi-D & further extensions
- Conclusions
Outline

• Introduction

• Well-balanced scheme for HydroStatic Equilibrium (HSE)
  • First order
  • Second order

• Multi-D & further extensions

• Conclusions
Evolution as a function of mass
i) Introduction

**Stellar life cycle**

**Evolution as a function of mass**

Where do the elements come from?

Solar system abundances

From primordial abundances of roughly H (75%), He (25%), and a very small amount of Li, elements are synthesized through various stages of stellar evolution, including nuclear burning and neutron capture. Adapted from Asplund 2005.
Stellar life cycle

Evolution as a function of mass

Core-Collapse Supernova

Birth  Life  Death

Protostar  Blue Supergiant  Red Giant  Type II Supernova  Neutron Star  Type Ia Supernova  White Dwarf  Planetary Nebula  Red Dwarf  White Dwarf

0.013 – 0.08 M☉  0.08 – 0.4 M☉  0.4 – 8 M☉  > 8 – 10 M☉  > 25 M☉  > 40 M☉
Core-collapse supernova

• General idea:
  • Implosion of iron core of massive \( M \gtrsim 8M_\odot \) at the end of thermonuclear evolution
  • Explosion powered by gravitational binding energy of forming compact remnant:

\[
E_b \approx 3 \times 10^{53} \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R}{10\text{km}} \right)^{-1} \text{ erg}
\]

GRAVITY BOMB!

\[ M \quad \text{Mass of remnant} \]
\[ R \quad \text{Radius of remnant} \]
i) Introduction

Core-collapse supernova

From S. Scheidegger

From M. Liebendoerfer
Radial profile

- The problem: (in our simulations)

Ability to maintain near hydrostatic equilibrium for a long time!
Outline

• Introduction

• Well-balanced scheme for HydroStatic Equilibrium (HSE)
  • First order
  • Second order

• Multi-D & further extensions

• Conclusions
Well-balanced scheme for HSE

- Consider 1D hydrodynamics eqs with gravity

\[
\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = S
\]

\[
\begin{bmatrix}
\rho \\
\rho v \\
E
\end{bmatrix}, \quad
\begin{bmatrix}
\rho v \\
\rho v^2 + p \\
(E + p)v
\end{bmatrix}, \quad
S = -\begin{bmatrix}
0 \\
\rho \\
\rho v
\end{bmatrix} \frac{\partial \phi}{\partial x}
\]

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)
Well-balanced scheme for HSE (2)

- Classical solution algorithm:

\[ u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right) + \Delta t S_{i}^{n} \]

- Numerical flux \( F_{i \pm 1/2}^{n} = \mathcal{F}(u_{i \pm 1/2}^{n,L}, u_{i \pm 1/2}^{n,R}) \) from (approximate) Riemann solver, e.g.

  - (Local) Lax-Friedrichs Lax (1954), Rusanov (1961)
  - HLL (C) Harten, Lax and van Leer (1983), Toro et al. (1994)
  - Roe Roe (1981)
Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

\[
\frac{\partial F}{\partial x} = S \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}
\]

EoS: \( p = p(\rho, e) \)
Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

\[
\frac{\partial F}{\partial x} = S \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}
\]

EoS: \( p = p(\rho, e) \)

Discretise in cells \([x_{i-1/2}, x_{i+1/2}]\)
ii) WB scheme for HSE

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:
\[
\frac{\partial F}{\partial x} = S \quad \implies \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}
\]

EoS: \( p = p(\rho, \epsilon) \)

Discretise in cells \( [x_{i-1/2}, x_{i+1/2}] \)

Define cell averages
\[
\begin{align*}
\mathbf{u}_i &= \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x, t^n) \, dx \\
S_i &= \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(\mathbf{u}(x, t)) \, dx
\end{align*}
\]
Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

\[
\frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) = S_i^n
\]

\[
F_{i+1/2}^{LxF} = \frac{1}{2} (F_i + F_{i+1}) - \frac{S_{\text{max}}}{2} (u_{i+1} - u_i)
\]

Contains also gravity induced gradient!

\[
F_{i-1/2}^{LxF} = \frac{1}{2} (F_{i-1} + F_i) - \frac{S_{\text{max}}}{2} (u_i - u_{i-1})
\]
ii) WB scheme for HSE

Well-balanced scheme for HSE (3)

Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \quad \rho(x) = \left[ \rho_0^{\frac{\gamma - 1}{\gamma}} - \frac{g}{K} \frac{\gamma - 1}{\gamma} x \right]^{\frac{1}{\gamma - 1}} \quad p = \frac{p_0}{\rho_0^{\gamma}} \rho^{\gamma} = K \rho^{\gamma}$$

$$x \in [0, 2]$$

Error in pressure:
(after 2 sound crossing times)

<table>
<thead>
<tr>
<th>N</th>
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<tbody>
<tr>
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<tr>
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$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

HLLC numerical flux
Well-balanced scheme for HSE (4)

• The problem: (in our simulations)

Ability to maintain near hydrostatic equilibrium for a long time!
ii) WB scheme for HSE

Well-balanced scheme for HSE (5)

- **Solutions:**
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(x) - u_0(r)$

Note: there are many, many more... especially for shallow-water eqs!!!
Well-balanced scheme for HSE (5)

• Solutions:
  - Define a **global** stationary state $u_0(r)$ at each time step and evolve $u(x) - u_0(r)$
  - Steady state preserving reconstructions, well-balanced schemes

Requirements

• Equilibrium not known in advance (self-gravity)
• Extensible for general EoS
• (At least) second order accuracy
• Preserve robustness of base shock capturing scheme

Note: there are many, many more... especially for shallow-water eqs!!!
Well-balanced scheme for HSE (6)

- Hydrostatic equilibrium

\[
\frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}
\]

Describes only a mechanical equilibrium...

Density and pressure not uniquely determined

\[ p = p(\rho, s) = p(\rho, T) \]

s  Entropy

T  Temperature

Arbitrary entropy or temperature profiles not (physically) stable (convection!)
Well-balanced scheme for HSE (7)

- Consider constant entropy profile

- Using the thermodynamic relation

\[ dh = T \, ds + \frac{dp}{\rho} \]

\[ h = e + \frac{p}{\rho} \quad \text{Enthalpy} \]

- Hydrostatic eq.

\[ \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial h}{\partial x} = - \frac{\partial \phi}{\partial x} \]

- Or simply

\[ h + \phi = \text{const} \]
ii) WB scheme for HSE

Well-balanced scheme for HSE (8)

Perform equilibrium reconstruction:

\[ h + \phi = \text{const} \]

Equilibrium enthalpy

\[ h_{0,i}(x) = h_i + \phi_i - \phi(x) \]

EoS

\[ h_{0,i}(x) = h\left(s_i, p_{0,i}(x)\right) \]

\[ p_{0,i}(x) \quad \& \quad \rho_{0,i}(x) \]

Eq. reconstructed primitive variables

\[ \mathbf{w}_{i\pm1/2}^n = \begin{bmatrix} \rho_{0,i}^n(x_{i\pm1/2}) \\ v_{x,i}^n \\ p_{0,i}^n(x_{i\pm1/2}) \end{bmatrix} \]
Well-balanced scheme for HSE (9)

• Well-balanced discretization of momentum source term

\[
S_{\rho v, i}^n = \frac{p_{0,i}^n(x_{i+1/2}) - p_{0,i}^n(x_{i-1/2})}{\Delta x} = - \int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + O(\Delta x^2)
\]

• Then for data satisfying \( h + \phi = \text{const} \), \( v_x = 0 \) and any consistent numerical flux

\[
\frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) = S_i^n
\]

Well-balanced wrt hydrostatic equilibrium!
ii) WB scheme for HSE

Well-balanced scheme for HSE (10)

- Second order extension:
  \[ r_{1,i}(x_j) = r_j - r_{0,i}(x_j) \]

  \[ r = \text{pressure, density} \quad \text{Eq. perturbation} \quad \text{Data} \quad \text{Equilibrium} \]

  Stencil: \( j = \ldots, i-1, i, i+1, \ldots \)

  \[ r_{1,i}(x) = r_{1,i}(x_i) + Dr_{1,i}(x - x_i) = \frac{r_{0,i}(x_{i-1}) - r_{i-1}}{\Delta x} + \frac{r_{i+1} - r_{0,i}(x_{i+1})}{\Delta x} \]

  Reconstruction in deviation from equilibrium

  Similar to Botta et al. 2004, Fuchs et al. 2010

- Time stepping:
  \[ u^* = u^n + \Delta t^n L(u^n) \]

  Strong Stability Preserving
  Runge-Kutta,
  Gottlieb et al. 2001

  \[ u^{**} = u^* + \Delta t^n L(u^*) \]

  \[ u^{n+1} = \frac{1}{2} (u^n + u^{**}) \]
Example 1

Hydrostatic atmosphere in a constant gravitational field

\[ \phi(x) = gx \quad \rho(x) = \left[ \rho_0^{\gamma - 1} - \frac{g}{K} \frac{\gamma - 1}{\gamma} x \right]^{\frac{1}{\gamma - 1}} \]

\[ p = \frac{p_0}{\rho_0} \rho^\gamma = K \rho^\gamma \]

\[ h + \phi = \text{const} \]

Error in pressure:

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</tr>
<tr>
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<td>1.1E-02 / 3.6E-14</td>
<td>1.6E-05 / 1.5E-14</td>
</tr>
<tr>
<td>512</td>
<td>5.3E-03 / 7.7E-14</td>
<td>4.1E-06 / 4.6E-14</td>
</tr>
<tr>
<td>1024</td>
<td>2.6E-03 / 5.7E-14</td>
<td>1.0E-06 / 6.1E-14</td>
</tr>
<tr>
<td>2048</td>
<td>1.3E-03 / 1.2E-13</td>
<td>2.6E-07 / 1.5E-14</td>
</tr>
<tr>
<td>rate</td>
<td>1.00 / -</td>
<td>2.00 / -</td>
</tr>
</tbody>
</table>

\[ Err = \frac{1}{N} \sum_i |p_i - p^0_i| \]
Example 2

Hydrostatic atmosphere in a constant gravitational field

\[ v(t, x = 0) = 10^{-6} \sin (8\pi t) \]

+ small perturbation

\[ N = 1024 \]

Error in pressure:

<table>
<thead>
<tr>
<th>N</th>
<th>2ndTVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>3.1E-05 / 1.9E-07</td>
</tr>
<tr>
<td>256</td>
<td>7.8E-06 / 6.8E-08</td>
</tr>
<tr>
<td>512</td>
<td>2.0E-06 / 2.5E-08</td>
</tr>
<tr>
<td>1024</td>
<td>4.8E-07 / 8.5E-09</td>
</tr>
<tr>
<td>2048</td>
<td>1.2E-07 / 4.1E-09</td>
</tr>
</tbody>
</table>

\[ Err = \frac{1}{N} \sum_{i} |p_i - p_i^0| \]
Example 3

Hydrostatic atmosphere in a constant gravitational field

ii) WB scheme for HSE

\[ v(t, x = 0) = 0.1 \sin (8\pi t) \]

- large perturbation

\[ N = 1024 \]

Error in pressure:

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>128</td>
<td>9.8E-03 / 1.1E-02</td>
</tr>
<tr>
<td>256</td>
<td>4.1E-03 / 4.9E-03</td>
</tr>
<tr>
<td>512</td>
<td>1.9E-03 / 2.0E-03</td>
</tr>
<tr>
<td>1024</td>
<td>8.7E-04 / 8.0E-04</td>
</tr>
<tr>
<td>2048</td>
<td>5.5E-04 / 3.3E-04</td>
</tr>
</tbody>
</table>

rate 1.05 / 1.28

\[ Err = \frac{1}{N} \sum_{i} |p_i - p_i^0| \]
Outline

• Introduction
• Well-balanced scheme for HydroStatic Equilibrium (HSE)
  • First order
  • Second order
• Multi-D & further extensions
• Conclusions
Multi-dimensional extension

- Straight forward directional application of HydroStatic Reconstruction

\[ \frac{d u_{i,j}}{dt} = L(u) = - \frac{1}{\Delta x} \left( F_{i+1/2,j} - F_{i-1/2,j} \right) - \frac{1}{\Delta y} \left( G_{i,j+1/2} - G_{i,j-1/2} \right) + S_{i,j} \]

- Hydrostatic equilibrium:

\[ h + \phi = const \]
Example 4

Polytrope: model star (e.g. main sequence stars, white dwarfs, neutron stars)

HSE: $\nabla p = -\rho \nabla \phi$  Poisson equation: $\nabla^2 \phi = -4\pi G \rho$

Equation of state $p = K \rho^\gamma$  $K = 1$

Take $\gamma = 2 \sim$ neutron stars

Then there's an exact solution:

$$\rho(x) = \rho_c \frac{\sin(\alpha r)}{r}$$

$$\phi(x) = -\gamma K \rho(x)$$

$$\alpha = \sqrt{\frac{2K}{4\pi G}} \quad r = \sqrt{x^2 + y^2 + z^2}$$
Example 4

Evolution for 20 “sound crossing” times

Error in density:

<table>
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<tbody>
<tr>
<td>32</td>
<td>1.3E-02 / 1.5E-14</td>
</tr>
<tr>
<td>64</td>
<td>3.6E-03 / 3.0E-14</td>
</tr>
<tr>
<td>128</td>
<td>1.0E-03 / 5.6E-14</td>
</tr>
</tbody>
</table>

rate 1.82 / -
Example 4

Small perturbation  \( \rho(x) = \rho_0(x) + Ae^{-100x^2} \quad A = 10^{-3} \)

<table>
<thead>
<tr>
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<th>2ndTVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>8.4E-04 / 1.1E-06</td>
</tr>
<tr>
<td>64</td>
<td>2.1E-04 / 3.7E-07</td>
</tr>
<tr>
<td>128</td>
<td>5.1E-05 / 1.1E-07</td>
</tr>
</tbody>
</table>

Error in pressure:

- NO HSE
- WITH HSE
- Reference

Pressure dev.
Example 4

Small perturbation: \( \rho(x) = \rho_0(x) + Ae^{-100x^2} \quad A = 10^{-3} \)

Error in velocity:

<table>
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</thead>
<tbody>
<tr>
<td>32</td>
<td>6.0E-04 / 8.8E-07</td>
</tr>
<tr>
<td>64</td>
<td>1.6E-04 / 3.0E-07</td>
</tr>
<tr>
<td>128</td>
<td>4.1E-05 / 8.5E-08</td>
</tr>
<tr>
<td>rate</td>
<td>1.92 / 1.69</td>
</tr>
</tbody>
</table>
ii) Multi-D & further extensions

Example 4

Large perturbation (detonation!)

Pressure

Error in pressure:

<table>
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<tbody>
<tr>
<td>32</td>
<td>3.3E-02 / 3.3E-02</td>
</tr>
<tr>
<td>64</td>
<td>1.8E-02 / 1.8E-02</td>
</tr>
<tr>
<td>128</td>
<td>9.6E-03 / 9.5E-03</td>
</tr>
<tr>
<td>rate</td>
<td>0.90 / 0.89</td>
</tr>
</tbody>
</table>
Example 4

ii) Multi-D & further extensions

Large perturbation (detonation!)

Velocity

Error in velocity:

<table>
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<tbody>
<tr>
<td>32</td>
<td>2.9E-02 / 2.8E-02</td>
</tr>
<tr>
<td>64</td>
<td>1.5E-02 / 1.4E-02</td>
</tr>
<tr>
<td>128</td>
<td>7.6E-03 / 7.6E-03</td>
</tr>
<tr>
<td>rate</td>
<td>0.96 / 0.94</td>
</tr>
</tbody>
</table>
Example 4

Rayleigh-Taylor instability

\[ N = 128 \]
Example 4

ii) Multi-D & further extensions

Rayleigh-Taylor instability

Rayleigh-Taylor “mushrooms”

$N = 128$
Well-balanced scheme for HSE

- Consider constant temperature profile

- Using the thermodynamic relation

  \[ dg = -s dT + \frac{dp}{\rho} \]

  \[ g = h - Ts \]

  Gibbs free energy

- Hydrostatic eq.

  \[ \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial g}{\partial x} = -\frac{\partial \phi}{\partial x} \]

- Or simply

  \[ g + \phi = \text{const} \]

  Reconstruction...
Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

\[
\frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) = S_i^n
\]

\[
\frac{\partial p}{\partial x} + O(\Delta x^2) = \frac{p_{i+1/2} - p_{i-1/2}}{\Delta x} = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)
\]

\[
\frac{(p_{i+1/2} - p_i) - (p_{i-1/2} - p_i)}{\Delta x} = -\frac{\rho_i}{2} \frac{(\phi_{i+1} - \phi_i) - (\phi_{i-1} - \phi_i)}{\Delta x}
\]
Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

\[
\frac{p_{i+1/2} - p_i}{\Delta x} - \frac{p_{i-1/2} - p_i}{\Delta x} = -\frac{\rho_i}{2} \left( \frac{\phi_{i+1} - \phi_i}{\Delta x} - \frac{\phi_{i-1} - \phi_i}{\Delta x} \right)
\]

Equilibrium reconstruction:

\[
p_{i+1/2} = p_i + \frac{\Delta x}{2} \Delta p_i^+
\]

\[
p_{i-1/2} = p_i - \frac{\Delta x}{2} \Delta p_i^-
\]

Equilibrium differences:

\[
\Delta p_i^+ = -\rho_i \frac{\phi_{i+1} - \phi_i}{\Delta x}
\]

\[
\Delta p_i^- = -\rho_i \frac{\phi_i - \phi_{i-1}}{\Delta x}
\]
Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

\[ p_{i+1/2}^L = p_{i+1/2}^R \]

\[ p_i + \frac{\Delta x}{2} \Delta p_i^+ = p_{i+1} - \frac{\Delta x}{2} \Delta p_{i+1}^- \]

\[ \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} \]

Discrete HydroStatic Equilibrium
Well-balanced scheme for HSE

Interested in **numerical** hydrostatic equilibrium:

$$p^L_{i+1/2} = p^R_{i+1/2}$$

**Equilibrium?**

**Requirement on Riemann solver:**

$$F^n_{i\pm 1/2} = \mathcal{F} \left( \begin{bmatrix} \rho^L_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix}, \begin{bmatrix} \rho^R_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ p_{i+1/2} \\ 0 \end{bmatrix}$$

e.g. HLLC, Roe

**Discrete HydroStatic Equilibrium**
Example 5

Hydrostatic atmosphere in a constant gravitational field

\[ \phi_i = g x_i \]
\[ \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} \]
\[ p_i = K_i \rho_i^\gamma \]

\[ x \in [0, 2] \]

\[ K = \begin{cases} 
2 & \text{if } x < 1 \\
1 & \text{if } x \geq 1 
\end{cases} \sim \text{entropy} \]

Error in pressure:

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\[ Err = \frac{1}{N} \sum_i |p_i - p_i^0| \]
Conclusions

- 1D well-balanced scheme for (isentropic, isothermal, arbitrary) hydrostatic equilibrium (for general EoS)

- Extension to higher-order?

- Non-zero velocity steady state?
  (e.g. for steady accretion flow)

- Multi-D well-balanced scheme for (isentropic, isothermal) hydrostatic equilibrium (for general EoS)

Thank you for your attention!!!